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A · SHORT · TABLE
OF
INTEGRALS

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A

SHORT TABLE OF INTEGRALS

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Since I cannot hope that these formulas are wholly free from misprints, I shall be grateful to any person who will call my attention to such errors as he may discover.

B. O. PEIRCE,
Harvard University, Cambridge.

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TABLE OF INTEGRALS.



I. FUNDAMENTAL FORMS.



1. $\int a \, dx = ax.$

2. $\int a f(x) \, dx = a \int f(x) \, dx.$

3. $\int \frac{dx}{x} = \log x. \quad [\log x = \log(-x) + (2k+1)\pi i.]$

4. $\int x^m \, dx = \frac{x^{m+1}}{m+1},$ when m is different from $-1.$

5. $\int e^x \, dx = e^x.$

6. $\int a^x \log a \, dx = a^x.$

7. $\int \frac{dx}{1+x^2} = \tan^{-1}x, \text{ or } -\operatorname{ctn}^{-1}x.$

8. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x, \text{ or } -\cos^{-1}x$

9. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x, \text{ or } -\operatorname{csc}^{-1}x.$

10. $\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{versin}^{-1}x, \text{ or } -\operatorname{coversin}^{-1}x.$

$$11. \int \cos x \, dx = \sin x, \text{ or } -\operatorname{coversin} x.$$

$$12. \int \sin x \, dx = -\cos x, \text{ or } \operatorname{versin} x.$$

$$13. \int \operatorname{ctn} x \, dx = \log \sin x.$$

$$14. \int \tan x \, dx = -\log \cos x.$$

$$15. \int \tan x \sec x \, dx = \sec x.$$

$$16. \int \sec^2 x \, dx = \tan x.$$

$$17. \int \csc^2 x \, dx = -\operatorname{ctn} x.$$

In the following formulas, u , v , w , and y represent any functions of x :

$$18. \int (u + v + w + \text{etc.}) \, dx = \int u \, dx + \int v \, dx + \int w \, dx + \text{etc.}$$

$$19 \text{ a. } \int u \, dv = uv - \int v \, du.$$

$$19 \text{ b. } \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx.$$

$$20. \int f(y) \, dx = \int \frac{f(y) \, dy}{\frac{dy}{dx}}$$

II. RATIONAL ALGEBRAIC FUNCTIONS.

A. — EXPRESSIONS INVOLVING $(a + bx)$.

The substitution of y or z for x , where $y \equiv a + bx$,
 $z \equiv (a + bx)/x$, gives

$$21. \int (a + bx)^m dx = \frac{1}{b} \int y^m dy.$$

$$22. \int x (a + bx)^m dx = \frac{1}{b^2} \int y^m (y - a) dy.$$

$$23. \int x^n (a + bx)^m dx = \frac{1}{b^{n+1}} \int y^m (y - a)^n dy.$$

$$24. \int \frac{x^n dx}{(a + bx)^m} = \frac{1}{b^{n+1}} \int \frac{(y - a)^n dy}{y^m}.$$

$$25. \int \frac{dx}{x^n (a + bx)^m} = -\frac{1}{a^{m+n-1}} \int \frac{(z - b)^{m+n-2} dz}{z^m}.$$

Whence

$$26. \int \frac{dx}{a + bx} = \frac{1}{b} \log (a + bx).$$

$$27. \int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}.$$

$$28. \int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}.$$

$$29. \int \frac{x dx}{a + bx} = \frac{1}{b^2} [a + bx - a \log (a + bx)].$$

$$30. \int \frac{x dx}{(a + bx)^2} = \frac{1}{b^2} \left[\log (a + bx) + \frac{a}{a + bx} \right].$$

$$31. \int \frac{x dx}{(a+bx)^3} = \frac{1}{b^2} \left[-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2} \right].$$

$$32. \int \frac{x^2 dx}{a+bx} = \frac{1}{b^3} \left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2 \log(a+bx) \right].$$

$$33. \int \frac{x^2 dx}{(a+bx)^2} = \frac{1}{b^3} \left[a+bx - 2a \log(a+bx) - \frac{a^2}{a+bx} \right].$$

$$34. \int \frac{dx}{x(a+bx)} = -\frac{1}{a} \log \frac{a+bx}{x}.*$$

$$35. \int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2} \log \frac{a+bx}{x}.$$

$$36. \int \frac{(a+bx) dx}{a'+b'x} = \frac{bx}{b'} + \frac{ab' - a'b}{b'^2} \log(a' + b'x).$$

$$37. \int (a+bx)^n (a'+b'x)^m dx = \frac{1}{(m+n+1)b} \left((a+bx)^{n+1} (a'+b'x)^m - m(ab' - a'b) \int (a+bx)^n (a'+b'x)^{m-1} dx \right).$$

$$\begin{aligned} 38. \int \frac{(a+bx)^n dx}{(a'+b'x)^m} &= -\frac{1}{(m-1)(ab' - a'b)} \left(\frac{(a+bx)^{n+1}}{(a'+b'x)^{m-1}} \right. \\ &\quad \left. + (m-n-2)b \int \frac{(a+bx)^n dx}{(a'+b'x)^{m-1}} \right) \\ &= -\frac{1}{(m-n-1)b'} \left(\frac{(a+bx)^n}{(a'+b'x)^{m-1}} \right. \\ &\quad \left. + n(ab' - a'b) \int \frac{(a+bx)^{n-1} dx}{(a'+b'x)^m} \right) \\ &= -\frac{1}{(m-1)b'} \left(\frac{(a+bx)^n}{(a'+b'x)^{m-1}} - nb \int \frac{(a+bx)^{n-1} dx}{(a'+b'x)^{m-1}} \right), \end{aligned}$$

$$* \int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2} \log \frac{a+bx}{x}.$$

$$39. \int \frac{dx}{(a+bx)(a'+b'x)} = \frac{1}{ab'-a'b} \cdot \log \frac{a'+b'x}{a+bx}.$$

$$40. \int \frac{dx}{(a+bx)^n (a'+b'x)^m} \\ = \frac{1}{(m-1)(ab'-a'b)} \left(\frac{1}{(a+bx)^{n-1} (a'+b'x)^{m-1}} \right. \\ \left. - (m+n-2)b \int \frac{dx}{(a+bx)^n (a'+b'x)^{m-1}} \right).$$

$$41. \int \frac{x dx}{(a+bx)(a'+b'x)} \\ = \frac{1}{ab'-a'b} \left(\frac{a}{b} \log(a+bx) - \frac{a'}{b'} \log(a'+b'x) \right).$$

$$42. \int \frac{dx}{(a+bx)^2 (a'+b'x)} \\ = \frac{1}{ab'-a'b} \left(\frac{1}{a+bx} + \frac{b'}{ab'-a'b} \log \frac{a'+b'x}{a+bx} \right).$$

$$43. \int \frac{x dx}{(a+bx)^2 (a'+b'x)} \\ = \frac{-a}{b(ab'-a'b)(a+bx)} - \frac{a'}{(ab'-a'b)^2} \log \frac{a'+b'x}{a+bx}.$$

$$44. \int \frac{x^2 dx}{(a+bx)^2 (a'+b'x)} = \frac{a^2}{b^2(ab'-a'b)(a+bx)} \\ + \frac{1}{(ab'-a'b)^2} \left[\frac{a'^2}{b'} \log(a'+b'x) + \frac{a(ab'-2a'b)}{b^2} \log(a+bx) \right].$$

$$45. \int (a+bx)^{\frac{1}{n}} dx = \frac{n}{(n+1)b} (a+bx)^{\frac{n+1}{n}}.$$

$$46. \int \frac{dx}{(a+bx)^{\frac{1}{n}}} = \frac{n}{(n-1)b} (a+bx)^{\frac{n-1}{n}}.$$

B. — EXPRESSIONS INVOLVING $(a + bx^n)$.

$$47. \int \frac{dx}{c^2 + x^2} = \frac{1}{c} \tan^{-1} \frac{x}{c} = \frac{1}{c} \sin^{-1} \frac{x}{\sqrt{x^2 + c^2}}.$$

$$48. \int \frac{dx}{c^2 - x^2} = \frac{1}{2c} \log \frac{c+x}{c-x}, \quad \int \frac{dx}{x^2 - c^2} = \frac{1}{2c} \log \frac{x-c}{x+c}.*$$

$$49. \int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \left(x \sqrt{\frac{b}{a}} \right), \text{ or } \frac{1}{\sqrt{-ab}} \cdot \tanh^{-1} \left(x \sqrt{\frac{-b}{a}} \right)$$

$$50. \int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}} \log \frac{\sqrt{a} + x\sqrt{-b}}{\sqrt{a} - x\sqrt{-b}}, \text{ if } a > 0, b < 0.$$

$$51. \int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a} \int \frac{dx}{a+bx^2}.$$

$$52. \int \frac{dx}{(a+bx^2)^{m+1}} = \frac{1}{2ma} \frac{x}{(a+bx^2)^m} + \frac{2m-1}{2ma} \int \frac{dx}{(a+bx^2)^m}$$

$$53. \int \frac{x dx}{a+bx^2} = \frac{1}{2b} \log \left(x^2 + \frac{a}{b} \right).$$

$$54. \int \frac{x dx}{(a+bx^2)^{m+1}} = \frac{1}{2} \int \frac{dz}{(a+bz)^{m+1}}, \text{ where } z = x^2.$$

$$55. \int \frac{dx}{x(a+bx^2)} = \frac{1}{2a} \log \frac{x^2}{a+bx^2}.$$

$$56. \int \frac{x^2 dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a+bx^2}.$$

$$57. \int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a} \int \frac{dx}{a+bx^2}.$$

$$58. \int \frac{x^2 dx}{(a+bx^2)^{m+1}} = \frac{-x}{2mb(a+bx^2)^m} + \frac{1}{2mb} \int \frac{dx}{(a+bx^2)^m}.$$

$$59. \int \frac{dx}{x^2(a+bx^2)^{m+1}} = \frac{1}{a} \int \frac{dx}{x^2(a+bx^2)^m} - \frac{b}{a} \int \frac{dx}{(a+bx^2)^{m+1}}.$$

* $\int \frac{dx}{c^2 - x^2} = \frac{1}{c} \tanh^{-1} \left(\frac{x}{c} \right); \quad \int \frac{dx}{x^2 - c^2} = -\frac{1}{c} \operatorname{ctnh}^{-1} \left(\frac{x}{c} \right).$

$$60. \int \frac{dx}{a+bx^3} = \frac{k}{3a} \left[\frac{1}{2} \log \left(\frac{(k+x)^2}{k^2-kx+x^2} \right) + \sqrt{3} \tan^{-1} \frac{2x-k}{k\sqrt{3}} \right], \text{ where } bk^3 = a.$$

$$61. \int \frac{x dx}{a+bx^3} = \frac{1}{3bk} \left[\frac{1}{2} \log \left(\frac{k^2-kx+x^2}{(k+x)^2} \right) + \sqrt{3} \tan^{-1} \frac{2x-k}{k\sqrt{3}} \right], \text{ where } bk^3 = a.$$

$$62. \int \frac{dx}{x(a+bx^n)} = \frac{1}{an} \log \frac{x^n}{a+bx^n}. \quad 63. \int \frac{dx}{(a+bx^n)^{m+1}} = \frac{1}{a} \int \frac{dx}{(a+bx^n)^m} - \frac{b}{a} \int \frac{x^n dx}{(a+bx^n)^{m+1}}.$$

$$64. \int \frac{x^m dx}{(a+bx^n)^{p+1}} = \frac{1}{b} \int \frac{x^{m-n} dx}{(a+bx^n)^p} - \frac{a}{b} \int \frac{x^{m-n} dx}{(a+bx^n)^{p+1}}.$$

$$65. \int \frac{dx}{x^m(a+bx^n)^{p+1}} = \frac{1}{a} \int \frac{dx}{x^m(a+bx^n)^p} - \frac{b}{a} \int \frac{dx}{x^{m-n}(a+bx^n)^{p+1}}.$$

$$66. \int x^{m-1}(a+bx^n)^p dx = \left\{ \begin{array}{l} \frac{1}{b(m+np)} \left[x^{m-n}(a+bx^n)^{p+1} - (m-n)a \int x^{m-n-1}(a+bx^n)^p dx \right] \\ \frac{1}{m+np} \left[x^m(a+bx^n)^p + npa \int x^{m-1}(a+bx^n)^{p-1} dx \right] \\ \frac{1}{ma} \left[x^m(a+bx^n)^{p+1} - (m+np+n)b \int x^{m+n-1}(a+bx^n)^p dx \right] \\ \frac{1}{an(p+1)} \left[-x^m(a+bx^n)^{p+1} + (m+np+n) \int x^{m-1}(a+bx^n)^{p+1} dx \right] \end{array} \right.$$

C. — EXPRESSIONS INVOLVING $(a + bx + cx^2)$.

Let $X = a + bx + cx^2$ and $q = 4ac - b^2$, then

$$67. \int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx + b}{\sqrt{q}}, \text{ or } -\frac{2}{\sqrt{-q}} \cdot \tanh^{-1} \frac{2cx + b}{\sqrt{-q}}$$

$$68. \int \frac{dx}{X} = \frac{1}{\sqrt{-q}} \log \frac{2cx + b - \sqrt{-q}}{2cx + b + \sqrt{-q}}, \text{ when } q < 0.$$

$$69. \int \frac{dx}{X^2} = \frac{2cx + b}{qX} + \frac{2c}{q} \int \frac{dx}{X}.$$

$$70. \int \frac{dx}{X^3} = \frac{2cx + b}{q} \left(\frac{1}{2X^2} + \frac{3c}{qX} \right) + \frac{6c^2}{q^2} \int \frac{dx}{X}.$$

$$71. \int \frac{dx}{X^{n+1}} = \frac{2cx + b}{nqX^n} + \frac{2(2n-1)c}{qn} \int \frac{dx}{X^n}.$$

$$72. \int \frac{x dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}.$$

$$73. \int \frac{x dx}{X^2} = -\frac{bx + 2a}{qX} - \frac{b}{q} \int \frac{dx}{X}.$$

$$74. \int \frac{x dx}{X^{n+1}} = -\frac{2a + bx}{nqX^n} - \frac{b(2n-1)}{nq} \int \frac{dx}{X^n}.$$

$$75. \int \frac{x^2}{X} dx = \frac{x}{c} - \frac{b}{2c^2} \log X + \frac{b^2 - 2ac}{2c^2} \int \frac{dx}{X}.$$

$$76. \int \frac{x^2}{X^2} dx = \frac{(b^2 - 2ac)x + ab}{cqX} + \frac{2a}{q} \int \frac{dx}{X}.$$

$$77. \int \frac{x^m dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n} - \frac{n-m+1}{2n-m+1} \cdot \frac{b}{c} \int \frac{x^{m-1} dx}{X^{n+1}} \\ + \frac{m-1}{2n-m+1} \cdot \frac{a}{c} \int \frac{x^{m-2} dx}{X^{n+1}}.$$

$$78. \int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}.$$

$$79. \int \frac{dx}{x^2X} = \frac{b}{2a^2} \log \frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a} \right) \int \frac{dx}{X}.$$

$$80. \int \frac{dx}{x^m X^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1} \cdot \frac{b}{a} \int \frac{dx}{x^{m-1}X^{n+1}} \\ - \frac{2n+m-1}{m-1} \cdot \frac{c}{a} \int \frac{dx}{x^{m-2}X^{n+1}}.$$

$$81. \int X^n dx = \frac{1}{2(2n+1)c} \left((b+2cx)X^n + nq \int X^{n-1} dx \right).$$

$$82. \int \frac{dx}{xX^n} = \frac{1}{2a(n-1)X^{n-1}} - \frac{b}{2a} \int \frac{dx}{X^n} + \frac{1}{a} \int \frac{dx}{xX^{n-1}}.$$

$$83. \int \frac{dx}{(a'+b'x)X} = \frac{1}{2(ab'^2 - a'bb' + a'^2c)} \left(b'(\log(a'+b'x))^2 \right. \\ \left. - \log X + (2a'e - bb') \int \frac{dx}{X} \right).$$

$$84. \int (a'+b'x)X^n dx = \frac{b'X^{n+1}}{2(n+1)c} + \frac{2a'e - bb'}{2c} \int X^n dx.$$

$$85. \int \frac{(a'+b'x)dx}{X^n} = -\frac{b'}{2(n-1)cX^{n-1}} + \frac{2a'e - bb'}{2c} \int \frac{dx}{X^n}.$$

$$86. \int (a'+b'x)^m X^n dx = \frac{1}{(m+2n+1)c} \left(b'(a'+b'x)^{m-1} X^{n+1} \right. \\ \left. + (m+n)(2a'e - bb') \int (a'+b'x)^{m-1} X^n dx \right. \\ \left. - (m-1)(ab'^2 - a'bb' + ca'^2) \int (a'+b'x)^{m-2} X^n dx \right).$$

$$\begin{aligned}
87. \int \frac{(a' + b'x)^m dx}{X^n} &= \frac{1}{q(n-1)} \left(\frac{(b + 2cx)(a' + b'x)^m}{X^{n-1}} \right. \\
&\quad - 2(m - 2n + 3)c \int \frac{(a' + b'x)^m dx}{X^{n-1}} \\
&\quad \left. + m(2a'c - bb') \int \frac{(a' + b'x)^{m-1} dx}{X^{n-1}} \right) \\
&= \frac{1}{(m - 2n + 1)c} \left(\frac{b'(a' + b'x)^{m-1}}{X^{n-1}} \right. \\
&\quad + (m - n)(2a'c - bb') \int \frac{(a' + b'x)^{m-1} dx}{X^n} \\
&\quad \left. - (m - 1)(ab'^2 - a'bb' + ca'^2) \int \frac{(a' + b'x)^{m-2} dx}{X^n} \right)
\end{aligned}$$

$$\begin{aligned}
88. \int \frac{X^n dx}{(a' + b'x)^m} &= \frac{1}{b'^2(m-1)} \left(\frac{-b'X^n}{(a' + b'x)^{m-1}} \right. \\
&\quad + n(bb' - 2a'c) \int \frac{X^{n-1} dx}{(a' + b'x)^{m-1}} \\
&\quad \left. + 2nc \int \frac{X^{n-1} dx}{(a' + b'x)^{m-2}} \right) \\
&= -\frac{1}{(m - 2n - 1)b'^2} \left(\frac{+b'X^n}{(a' + b'x)^{m-1}} \right. \\
&\quad + 2b'n(ab'^2 - a'bb' + ca'^2) \int \frac{X^{n-1} dx}{(a' + b'x)^m} \\
&\quad \left. + n(bb' - 2a'c) \int \frac{X^{n-1} dx}{(a' + b'x)^{m-1}} \right).
\end{aligned}$$

$$\begin{aligned}
89. \int \frac{dx}{(a' + b'x)^m X^n} &= -\frac{1}{(m-1)(ab'^2 - a'bb' + ca'^2)} \left(\frac{b'}{(a' + b'x)^{m-1} X^{n-1}} \right. \\
&\quad + (m+n-2)(bb' - 2ca') \int \frac{dx}{(a' + b'x)^{m-1} X^n} \\
&\quad \left. + (m+2n-3)c \int \frac{dx}{(a' + b'x)^{m-2} X^n} \right) \\
&= \frac{1}{2(ab'^2 - a'bb' + ca'^2)} \left(\frac{b'}{(n-1)(a' + b'x)^{m-1} X^{n-1}} \right. \\
&\quad + (2a'c - bb') \int \frac{dx}{(a' + b'x)^{m-1} X^n} \\
&\quad \left. + \frac{(m+2n-3)b'^2}{n+1} \int \frac{dx}{(a' + b'x)^m X^{n-1}} \right).
\end{aligned}$$

If $ab'^2 - a'bb' + ca'^2 = 0$,

$$\begin{aligned}
\int \frac{dx}{(a' + b'x)^m X^n} &= \frac{-1}{(m+n-1)(bb' - 2a'c)} \left(\frac{b'}{(a' + b'x)^m X^{n-1}} \right. \\
&\quad \left. + (m+2n-2)c \int \frac{dx}{(a' + b'x)^{m-1} X^n} \right).
\end{aligned}$$

D. — RATIONAL FRACTIONS.

Every proper fraction can be represented by the general form :

$$\frac{f(x)}{F(x)} = \frac{g_1 x^{n-1} + g_2 x^{n-2} + g_3 x^{n-3} + \dots + g_n}{x^n + k_1 x^{n-1} + k_2 x^{n-2} + \dots + k_n}.$$

If a, b, c , etc., are the roots of the equation $F(x) = 0$, so that

$$F(x) = (x-a)^p (x-b)^q (x-c)^r \dots,$$

then

$$\begin{aligned} \frac{f(x)}{F(x)} &= \frac{A_1}{(x-a)^p} + \frac{A_2}{(x-a)^{p-1}} + \frac{A_3}{(x-a)^{p-2}} + \cdots + \frac{A_p}{x-a} \\ &+ \frac{B_1}{(x-b)^q} + \frac{B_2}{(x-b)^{q-1}} + \frac{B_3}{(x-b)^{q-2}} + \cdots + \frac{B_q}{x-b} \\ &+ \frac{C_1}{(x-c)^r} + \frac{C_2}{(x-c)^{r-1}} + \frac{C_3}{(x-c)^{r-2}} + \cdots + \frac{C_r}{x-c} \\ &+ \quad \cdots \quad \quad \quad \cdots \quad \quad \quad \cdots \quad \quad \quad \cdots \quad \quad \cdots, \end{aligned}$$

where the numerators of the separate fractions may be determined by the equations

$$A_m = \frac{\phi_1^{[m-1]}(a)}{(m-1)!}, \quad B_m = \frac{\phi_2^{[m-1]}(b)}{(m-1)!} \quad \text{etc., etc.}$$

$$\phi_1(x) = \frac{f(x)(x-a)^p}{F(x)}, \quad \phi_2(x) = \frac{f(x)(x-b)^q}{F(x)}, \quad \text{etc., etc.}$$

If a, b, c , etc., are single roots, then $p = q = r = \cdots = 1$, and

$$\frac{f(x)}{F(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \cdots$$

where $A = \frac{f(a)}{F'(a)}, \quad B = \frac{f(b)}{F'(b)}, \quad \text{etc.}$

The simpler fractions, into which the original fraction is thus divided, may be integrated by means of the formulas :

$$90. \int \frac{h dx}{(mx+n)^l} = \int \frac{h d(mx+n)}{m(mx+n)^l} = \frac{h}{m(1-l)(mx+n)^{l-1}},$$

$$\text{and} \quad \int \frac{h dx}{mx+n} = \frac{h}{m} \log (mx+n).$$

If any of the roots of the equation $f(x) = 0$ are imaginary, the parts of the integral which arise from conjugate roots can be combined and the integral brought into a real form. The following formula, in which $i = \sqrt{-1}$, is often useful in combining logarithms of conjugate complex quantities:

$$\log(x \pm yi) = \frac{1}{2} \log(x^2 + y^2) \pm i \tan^{-1} \frac{y}{x}.$$

The identities given below are sometimes convenient:

$$\frac{1}{(a + bx^2)(a' + b'x^2)} \equiv \frac{1}{a'b - ab'} \cdot \left[\frac{b}{a + bx^2} - \frac{b'}{a' + b'x^2} \right],$$

$$\frac{m + nx}{(k + lx)(a + bx + cx^2)} \equiv \frac{1}{al^2 + ck^2 - bkl}.$$

$$\left[\frac{l(ml - nk)}{k + lx} + \frac{c(nk - ml)x + (aln + ck m - blm)}{a + bx + cx^2} \right],$$

$$\frac{l + mx^n}{(a + bx^n)(a' + b'x^n)} \equiv \frac{1}{a'b - ab'} \cdot \left[\frac{bl - am}{a + bx^n} + \frac{a'm - b'l}{a' + b'x^n} \right].$$

$$\frac{1}{(x + a)(x + b)(x + c)} = \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c},$$

where

$$A = \frac{1}{(a - b)(a - c)}, \quad B = \frac{1}{(b - c)(b - a)}, \quad C = \frac{1}{(c - a)(c - b)}.$$

$$\frac{1}{(x + a)(x + b)(x + c)(x + g)} = \frac{A}{x + a} + \frac{B}{x + b} + \frac{C}{x + c} + \frac{G}{x + g},$$

where

$$A = \frac{1}{(b - a)(c - a)(g - a)}, \quad B = \frac{1}{(a - b)(c - b)(g - b)}, \text{ etc.}$$

III. IRRATIONAL ALGEBRAIC FUNCTIONS.

A. — EXPRESSIONS INVOLVING $\sqrt{a+bx}$.

The substitution of a new variable of integration, $y = \sqrt{a+bx}$, gives

$$91. \int \sqrt{a+bx} dx = \frac{2}{3b} \sqrt{(a+bx)^3}.$$

$$92. \int x \sqrt{a+bx} dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$$

$$93. \int x^2 \sqrt{a+bx} dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}$$

$$94. \int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{dx}{x\sqrt{a+bx}}.$$

$$95. \int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$$

$$96. \int \frac{x dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2} \sqrt{a+bx}.$$

$$97. \int \frac{x^2 dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3} \sqrt{a+bx}.$$

$$98. \int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right), \text{ for } a > 0.$$

$$99. \int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bx}{-a}}, \text{ or } \frac{-2}{\sqrt{a}} \cdot \tanh^{-1} \sqrt{\frac{a+bx}{a}}.$$

$$100. \int \frac{dx}{x^2 \sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a} \int \frac{dx}{x \sqrt{a+bx}}.$$

$$101. \int (a+bx)^{\pm \frac{n}{2}} dx = \frac{2}{b} \int y^{1 \pm n} dy = \frac{2(a+bx)^{\frac{2 \pm n}{2}}}{b(2 \pm n)}.$$

$$102. \int x(a+bx)^{\pm \frac{n}{2}} dx = \frac{2}{b^2} \left[\frac{(a+bx)^{\frac{4 \pm n}{2}}}{4 \pm n} - \frac{a(a+bx)^{\frac{2 \pm n}{2}}}{2 \pm n} \right].$$

$$103. \int \frac{x^m dx}{\sqrt{a+bx}} = \frac{2x^m \sqrt{a+bx}}{(2m+1)b} - \frac{2ma}{(2m+1)b} \int \frac{x^{m-1} dx}{\sqrt{a+bx}}.$$

$$104. \int \frac{dx}{x^n \sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a} \int \frac{dx}{x^{n-1} \sqrt{a+bx}}$$

$$105. \int \frac{(a+bx)^{\frac{n}{2}} dx}{x} = b \int (a+bx)^{\frac{n-2}{2}} dx + a \int \frac{(a+bx)^{\frac{n-2}{2}}}{x} dx.$$

$$106. \int \frac{dx}{x(a+bx)^{\frac{m}{2}}} = \frac{1}{a} \int \frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a} \int \frac{dx}{(a+bx)^{\frac{m}{2}}}.$$

$$107. \int f(x, \sqrt[n]{a+bx}) dx = \frac{n}{b} \int f\left(\frac{z^n - a}{b}, z\right) z^{n-1} dz,$$

where $z^n = a + bx$.

$$108. \int (a+bx)^{\frac{m}{n}} dx = \frac{n(a+bx)^{\frac{m+n}{n}}}{b(m+n)}.$$

$$109. \int f(x, (a+bx)^{\frac{m}{n}}, (a+bx)^{\frac{p}{q}}, \dots) dx \\ = \frac{s}{b} \int f\left(\frac{y^s - a}{b}, y^{\frac{ms}{n}}, y^{\frac{ps}{q}}, \dots\right) y^{s-1} dy,$$

where $y^s = a + bx$, and s is the least common multiple of n , q , etc.

B.—EXPRESSIONS INVOLVING BOTH $\sqrt{a+bx}$ AND $\sqrt{a'+b'x}$.

Let $u = a + bx$, $v = a' + b'x$, and $k = ab' - a'b$, then

$$110. \int \sqrt{uv} \, dx = \frac{k + 2bv}{4bb'} \sqrt{uv} - \frac{k^2}{8bb'} \int \frac{dx}{\sqrt{uv}}.$$

$$111. \int \frac{\sqrt{v} \, dx}{\sqrt{u}} = \frac{1}{b} \sqrt{uv} - \frac{k}{2b} \int \frac{dx}{\sqrt{uv}}.$$

$$112. \int \frac{x \, dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bb'} - \frac{ab' + a'b}{2bb'} \int \frac{dx}{\sqrt{uv}}.$$

$$\begin{aligned} 113. \int \frac{dx}{\sqrt{uv}} &= \frac{2}{\sqrt{bb'}} \log(\sqrt{bb'u} + b\sqrt{v}) \\ &= \frac{2}{\sqrt{-bb'}} \tan^{-1} \sqrt{\frac{-b'u}{bv}}, \text{ or } \frac{2}{\sqrt{bb'}} \tanh^{-1} \sqrt{\frac{b'u}{bv}} \\ &= \frac{1}{\sqrt{-bb'}} \sin^{-1} \frac{2bb'x + a'b + ab'}{k}. \end{aligned}$$

$$114. \int \frac{dx}{v\sqrt{u}} = \frac{1}{\sqrt{kb'}} \log \frac{b'\sqrt{u} - \sqrt{kb'}}{b'\sqrt{u} + \sqrt{kb'}} = \frac{2}{\sqrt{-kb'}} \tan^{-1} \frac{b'\sqrt{u}}{\sqrt{-kb'}}.$$

$$115. \int \frac{dx}{v\sqrt{uv}} = -\frac{2\sqrt{u}}{k\sqrt{v}}.$$

$$116. \int v^m \sqrt{u} \, dx = \frac{1}{(2m+3)b'} \left(2v^{m+1} \sqrt{u} + k \int \frac{v^m dx}{\sqrt{u}} \right).$$

$$\begin{aligned} 117. \int \frac{\sqrt{u} \, dx}{v^m} &= -\frac{1}{(2m-3)b'} \left(\frac{2\sqrt{u}}{v^{m-1}} + k \int \frac{dx}{v^m \sqrt{u}} \right) \\ &= \frac{1}{(m-1)b'} \left(-\frac{\sqrt{u}}{v^{m-1}} + \frac{1}{2} b \int \frac{dx}{v^{m-1} \sqrt{u}} \right). \end{aligned}$$

$$118. \int \frac{v^m dx}{\sqrt{u}} = \frac{2}{(2m+1)b} \left(v^m \sqrt{u} - mk \int \frac{v^{m-1} dx}{\sqrt{u}} \right).$$

$$119. \int \frac{dx}{v^m \sqrt{u}} = -\frac{1}{(m-1)k} \left(\frac{\sqrt{u}}{v^{m-1}} + (m - \frac{3}{2})b \int \frac{dx}{v^{m-1} \sqrt{u}} \right).$$

$$120. \int v^m u^{n-\frac{1}{2}} dx = \frac{1}{(2m+2n+1)b'} \left(2 v^{m+1} u^{n-\frac{1}{2}} \right. \\ \left. + (2n-1)k \int v^m u^{n-\frac{3}{2}} dx \right).$$

$$121. \int v^m u^{-(n+\frac{1}{2})} dx = \frac{1}{(2n-1)k} \left(2 v^{m+1} u^{-(n-\frac{1}{2})} \right. \\ \left. - (2m-2n+3)b' \int v^m u^{-(n-\frac{1}{2})} dx \right) \\ = \frac{2}{(2n-1)b} \left(-v^m u^{-(n-\frac{1}{2})} \right. \\ \left. + mb' \int v^{m-1} u^{-(n-\frac{1}{2})} dx \right).$$

$$122. \int v^{-m} u^{(n-\frac{1}{2})} dx = \frac{-1}{(2m-2n-1)b'} \left(2 u^{n-\frac{1}{2}} v^{-(m-1)} \right. \\ \left. + (2n-1)k \int u^{n-\frac{3}{2}} v^{-m} dx \right) \\ = \frac{1}{(m-1)b'} \left(-u^{n-\frac{1}{2}} v^{-(m-1)} \right. \\ \left. + (n-\frac{1}{2})b \int u^{n-\frac{3}{2}} v^{-(m-1)} dx \right).$$

$$123. \int v^{-m} u^{-(n+\frac{1}{2})} dx = \frac{1}{(2n-1)k} \left(2 v^{-(m-1)} u^{-(n-\frac{1}{2})} \right. \\ \left. + (2m+2n-3)b' \int v^{-m} u^{-(n-\frac{1}{2})} dx \right).$$

C. — EXPRESSIONS INVOLVING $\sqrt{x^2 \pm a^2}$ AND $\sqrt{a^2 - x^2}$.

$$124. \int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} [x \sqrt{x^2 \pm a^2} \pm a^2 \log (x + \sqrt{x^2 \pm a^2})].^*$$

$$125. \int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left(x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right).$$

$$126. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log (x + \sqrt{x^2 \pm a^2}).^*$$

$$127. \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}, \text{ or } -\cos^{-1} \frac{x}{a}.$$

$$128. \int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \cos^{-1} \frac{a}{x}, \text{ or } \frac{1}{a} \sec^{-1} \frac{x}{a}.$$

$$129. \int \frac{dx}{x \sqrt{a^2 \pm x^2}} = -\frac{1}{a} \log \left(\frac{a + \sqrt{a^2 \pm x^2}}{x} \right).^*$$

$$130. \int \frac{\sqrt{a^2 \pm x^2}}{x} dx = \sqrt{a^2 \pm x^2} - a \log \frac{a + \sqrt{a^2 \pm x^2}}{x}.^*$$

$$131. \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cos^{-1} \frac{a}{x}.$$

$$132. \int \frac{x dx}{\sqrt{a^2 \pm x^2}} = \pm \sqrt{a^2 \pm x^2}.$$

$$133. \int \frac{x dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}.$$

$$134. \int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} \sqrt{(x^2 \pm a^2)^3}.$$

$$135. \int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} \sqrt{(a^2 - x^2)^3}.$$

$$\begin{aligned} \bullet \log \left(\frac{x + \sqrt{x^2 + a^2}}{a} \right) &= \sinh^{-1} \left(\frac{x}{a} \right); \log \left(\frac{x + \sqrt{x^2 - a^2}}{a} \right) = \cosh^{-1} \left(\frac{x}{a} \right); \\ \log \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right) &= \operatorname{sech}^{-1} \left(\frac{x}{a} \right); \log \left(\frac{a + \sqrt{a^2 + x^2}}{x} \right) = \operatorname{csech}^{-1} \left(\frac{x}{a} \right). \end{aligned}$$

$$\begin{aligned}
 136. \int \sqrt{(x^2 \pm a^2)^3} dx \\
 = \frac{1}{4} \left[x \sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2} \sqrt{x^2 \pm a^2} + \frac{3a^4}{2} \log(x + \sqrt{x^2 \pm a^2}) \right]^*
 \end{aligned}$$

$$\begin{aligned}
 137. \int \sqrt{(a^2 - x^2)^3} dx \\
 = \frac{1}{4} \left[x \sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2} \sqrt{a^2 - x^2} + \frac{3a^4}{2} \sin^{-1} \frac{x}{a} \right].
 \end{aligned}$$

$$138. \int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}}.$$

$$139. \int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2 \sqrt{a^2 - x^2}}.$$

$$140. \int \frac{x dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}.$$

$$141. \int \frac{x dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}.$$

$$142. \int x \sqrt{(x^2 \pm a^2)^3} dx = \frac{1}{5} \sqrt{(x^2 \pm a^2)^5}.$$

$$143. \int x \sqrt{(a^2 - x^2)^3} dx = -\frac{1}{5} \sqrt{(a^2 - x^2)^5}.$$

$$\begin{aligned}
 144. \int x^2 \sqrt{x^2 \pm a^2} dx \\
 = \frac{x}{4} \sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8} x \sqrt{x^2 \pm a^2} - \frac{a^4}{8} \log(x + \sqrt{x^2 \pm a^2}).
 \end{aligned}$$

$$\begin{aligned}
 145. \int x^2 \sqrt{a^2 - x^2} dx \\
 = -\frac{x}{4} \sqrt{(a^2 - x^2)^3} + \frac{a^2}{8} \left(x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right).
 \end{aligned}$$

$$\log z = \sinh^{-1} \left(\frac{z^2 - 1}{2z} \right) = \cosh^{-1} \left(\frac{z^2 + 1}{2z} \right); \quad \tanh^{-1} z = -i \cdot \tan^{-1} \frac{z}{i}.$$

* (See Note on pages 20-21.)

$$146. \int \frac{\sqrt{a^2 \pm x^2} dx}{x^3} = -\frac{\sqrt{a^2 \pm x^2}}{2x^2} \pm \frac{1}{2} \int \frac{dx}{x\sqrt{a^2 \pm x^2}}.$$

$$147. \int x^3 \sqrt{a^2 \pm x^2} dx = (\pm \frac{1}{5} x^2 - \frac{1}{15} a^2) \sqrt{(a^2 \pm x^2)^3}.$$

$$148. \int \frac{dx}{x^3 \sqrt{a^2 \pm x^2}} = -\frac{\sqrt{a^2 \pm x^2}}{2a^2 x^2} \mp \frac{1}{2a^2} \int \frac{dx}{x\sqrt{a^2 \pm x^2}}.$$

$$149. \int \frac{dx}{x^3 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2 x^2} + \frac{1}{2a^3} \sec^{-1} \left(\frac{x}{a} \right).$$

$$150. \int \frac{x^2 dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2} \sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \log (x + \sqrt{x^2 \pm a^2}).^*$$

$$151. \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}.$$

$$152. \int \frac{dx}{x^2 \sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}.$$

$$153. \int \frac{dx}{x^2 \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x}.$$

$$154. \int \frac{\sqrt{x^2 \pm a^2} dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log (x + \sqrt{x^2 \pm a^2}).^*$$

$$155. \int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a}.$$

$$156. \int \frac{x^2 dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log (x + \sqrt{x^2 \pm a^2}).^*$$

$$157. \int \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{a}.$$

* (See Note on pages 20-21.)

$$158. \int \frac{f(x^2) dx}{\sqrt{a+cx^2}} = g \int f\left(\frac{au^2}{g^2-cu^2}\right) \frac{du}{(g^2-cu^2)},$$

$$\text{where } u = \frac{gx}{\sqrt{a+cx^2}}.$$

$$159. \int \frac{xf(x^2) dx}{\sqrt{a+cx^2}} = \frac{1}{c} \int f\left(\frac{u^2-a}{c}\right) du, \text{ where } u^2 = a+cx^2.$$

D. — EXPRESSIONS INVOLVING $\sqrt{a+bx+cx^2}$.

Let $X = a + bx + cx^2$, $q = 4ac - b^2$, and $k = \frac{4c}{q}$. In order to rationalize the function $f(x, \sqrt{a+bx+cx^2})$ we may put $\sqrt{a+bx+cx^2} = \sqrt{\pm c} \sqrt{A+Bx \pm x^2}$, according as c is positive or negative, and then substitute for x a new variable z , such that

$$z = \sqrt{A+Bx+x^2} \pm x, \text{ if } c > 0.$$

$$z = \frac{\sqrt{A+Bx-x^2} - \sqrt{A}}{x}, \text{ if } c < 0 \text{ and } \frac{a}{-c} > 0.$$

$$z = \sqrt{\frac{x-\beta}{a-x}}, \text{ where } a \text{ and } \beta \text{ are the roots of the equation}$$

$$A+Bx-x^2=0, \text{ if } c < 0 \text{ and } \frac{a}{-c} < 0.$$

By rationalization, or by the aid of reduction formulas, may be obtained the values of the following integrals:

$$160. \int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}} \log \left(\sqrt{X} + x\sqrt{c} + \frac{b}{2\sqrt{c}} \right), \text{ if } c > 0.$$

$$161. \int \frac{dx}{\sqrt{X}} = \frac{-1}{\sqrt{-c}} \sin^{-1} \left(\frac{2cx+b}{\sqrt{-q}} \right), \text{ or } \frac{1}{\sqrt{c}} \sinh^{-1} \left(\frac{2cx+b}{\sqrt{q}} \right).$$

$$162. \int \frac{dx}{X\sqrt{X}} = \frac{2(2cx+b)}{q\sqrt{X}}.$$

$$163. \int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx+b)}{3q\sqrt{X}} \left(\frac{1}{X} + 2k \right).$$

$$164. \int \frac{dx}{X^n\sqrt{X}} = \frac{2(2cx+b)\sqrt{X}}{(2n-1)qX^n} + \frac{2k(n-1)}{2n-1} \int \frac{dx}{X^{n-1}\sqrt{X}}.$$

$$165. \int \sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k} \int \frac{dx}{\sqrt{X}}.$$

$$166. \int X\sqrt{X} dx = \frac{(2cx+b)\sqrt{X}}{8c} \left(X + \frac{3}{2k} \right) + \frac{3}{8k^2} \int \frac{dx}{\sqrt{X}}.$$

$$167. \int X^2\sqrt{X} dx \\ = \frac{(2cx+b)\sqrt{X}}{12c} \left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2} \right) + \frac{5}{16k^3} \int \frac{dx}{\sqrt{X}}.$$

$$168. \int X^n\sqrt{X} dx = \frac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \frac{2n+1}{2(n+1)k} \int \frac{X^n dx}{\sqrt{X}}.$$

$$169. \int \frac{x dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{X}}.$$

$$170. \int \frac{x dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}.$$

$$171. \int \frac{x dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c} \int \frac{dx}{X^n\sqrt{X}}.$$

$$172. \int \frac{x^2 dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2} \right) \sqrt{X} + \frac{3b^2-4ac}{8c^2} \int \frac{dx}{\sqrt{X}}.$$

$$173. \int \frac{x^2 dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c} \int \frac{dx}{\sqrt{X}}.$$

$$174. \int \frac{x^2 dx}{X^n \sqrt{X}} = \frac{(2b^2 - 4ac)x + 2ab}{(2n-1)cq X^{n-1} \sqrt{X}} + \frac{4ac + (2n-3)b^2}{(2n-1)cq} \int \frac{dx}{X^{n-1} \sqrt{X}}.$$

$$175. \int \frac{x^3 dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2} \right) \sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^3}{16c^3} \right) \int \frac{dx}{\sqrt{X}}.$$

$$176. \int x \sqrt{X} dx = \frac{X\sqrt{X}}{3c} - \frac{b}{2c} \int \sqrt{X} dx.$$

$$177. \int xX\sqrt{X} dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c} \int X\sqrt{X} dx.$$

$$178. \int \frac{xX^n dx}{\sqrt{X}} = \frac{X^n\sqrt{X}}{(2n+1)c} - \frac{b}{2c} \int \frac{X^n dx}{\sqrt{X}}.$$

$$179. \int x^2\sqrt{X} dx = \left(x - \frac{5b}{6c} \right) \frac{X\sqrt{X}}{4c} + \frac{5b^2 - 4ac}{16c^2} \int \sqrt{X} dx.$$

$$180. \int \frac{x^2X^n dx}{\sqrt{X}} = \frac{xX^n\sqrt{X}}{2(n+1)c} - \frac{(2n+3)b}{4(n+1)c} \int \frac{xX^n dx}{\sqrt{X}} - \frac{a}{2(n+1)c} \int \frac{X^n dx}{\sqrt{X}}.$$

$$181. \int x^3\sqrt{X} dx = \left(x^2 - \frac{7bx}{8c} + \frac{35b^2}{48c^2} - \frac{2a}{3c} \right) \frac{X\sqrt{X}}{5c} + \left(\frac{3ab}{8c^2} - \frac{7b^3}{32c^3} \right) \int \sqrt{X} dx.$$

$$182. \int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}} \log \left(\frac{\sqrt{X} + \sqrt{a}}{x} + \frac{b}{2\sqrt{a}} \right), \text{ if } a > 0.$$

$$183. \int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}} \sin^{-1} \left(\frac{bx+2a}{x\sqrt{-q}} \right), \text{ or } \frac{-1}{\sqrt{a}} \sinh^{-1} \frac{2a+bx}{x\sqrt{q}}.$$

$$184. \int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}, \text{ if } a=0.$$

$$185. \int \frac{dx}{xX^n\sqrt{X}} \\ = \frac{\sqrt{X}}{(2n-1)aX^n} + \frac{1}{a} \int \frac{dx}{xX^{n-1}\sqrt{X}} - \frac{b}{2a} \int \frac{dx}{X^n\sqrt{X}}.$$

$$186. \int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{X}}.$$

$$187. \int \frac{\sqrt{X}dx}{x} = \sqrt{X} + \frac{b}{2} \int \frac{dx}{\sqrt{X}} + a \int \frac{dx}{x\sqrt{X}}.$$

$$188. \int \frac{X^n dx}{x\sqrt{X}} = \frac{X^n}{(2n-1)\sqrt{X}} + a \int \frac{X^{n-1}dx}{x\sqrt{X}} + \frac{b}{2} \int \frac{X^{n-1}dx}{\sqrt{X}}.$$

$$189. \int \frac{\sqrt{X}dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{X}} + a \int \frac{dx}{\sqrt{X}}.$$

$$190. \int \frac{x^m dx}{X^n\sqrt{X}} = \frac{1}{c} \int \frac{x^{m-2}dx}{X^{n-1}\sqrt{X}} - \frac{b}{c} \int \frac{x^{m-1}dx}{X^n\sqrt{X}} - \frac{a}{c} \int \frac{x^{m-2}dx}{X^n\sqrt{X}}.$$

$$191. \int \frac{x^m X^n dx}{\sqrt{X}} = \frac{x^{m-1} X^n \sqrt{X}}{(2n+m)c} - \frac{(2n+2m-1)b}{2c(2n+m)} \int \frac{x^{m-1} X^n dx}{\sqrt{X}} \\ - \frac{(m-1)a}{(2n+m)c} \int \frac{x^{m-2} X^n dx}{\sqrt{X}}.$$

$$192. \int \frac{dx}{x^m X^n \sqrt{X}} \\ = -\frac{\sqrt{X}}{(m-1)a x^{m-1} X^n} - \frac{(2n+2m-3)b}{2a(m-1)} \int \frac{dx}{x^{m-1} X^n \sqrt{X}} \\ - \frac{(2n+m-2)c}{(m-1)a} \int \frac{dx}{x^{m-2} X^n \sqrt{X}}.$$

$$193. \int \frac{X^n dx}{x^m \sqrt{X}} = -\frac{X^{n-1} \sqrt{X}}{(m-1)x^{m-1}} + \frac{(2n-1)b}{2(m-1)} \int \frac{X^{n-1} dx}{x^{m-1} \sqrt{X}} \\ + \frac{(2n-1)c}{m-1} \int \frac{X^{n-1} dx}{x^{m-2} \sqrt{X}}.$$

$$194. \int f(x, \sqrt{(x-a)(x-b)}) dx \\ = 2(a-b) \int f \left\{ \frac{bu^2-a}{u^2-1}, \frac{u(b-a)}{u^2-1} \right\} \frac{u du}{(u^2-1)^2},$$

where $u^2(x-b) = x-a$.

E. — EXPRESSIONS INVOLVING PRODUCTS OF POWERS OF
 $(a' + b'x)$ AND $\sqrt{a + bx + cx^2}$.

Let $X = a + bx + cx^2$, $v = a' + b'x$, $q = 4ac - b^2$,
 $\beta = bb' - 2a'c$, $k = ab'^2 - a'bb' + ca'^2$, then

$$195. \int \frac{dx}{v \sqrt{X}} = \frac{1}{\sqrt{k}} \log \frac{2k + \beta v - 2b' \sqrt{kX}}{v} \\ = \frac{1}{\sqrt{-k}} \tan^{-1} \frac{2k + \beta v}{2b' \sqrt{-kX}} \\ = \frac{1}{\sqrt{-k}} \sin^{-1} \frac{2k + \beta v}{b'v \sqrt{-q}}, \text{ if } k \neq 0.$$

$$196. \int \frac{dx}{v \sqrt{X}} = -\frac{2b' \sqrt{X}}{\beta v}, \text{ if } k = 0: \\ \text{thus, } \int \frac{dx}{(x \pm 1) \sqrt{x^2 - 1}} = \pm \sqrt{\frac{x \mp 1}{x \pm 1}}.$$

$$197. \int \frac{dx}{v^2 \sqrt{X}} = -\frac{b' \sqrt{X}}{kv} - \frac{\beta}{2k} \int \frac{dx}{v \sqrt{X}}.$$

$$198. \int \frac{dx}{v^3 \sqrt{X}} = -\frac{2b' \sqrt{X}}{3\beta v^2} - \frac{2c}{3\beta} \int \frac{dx}{v \sqrt{X}}, \text{ if } k = 0.$$

$$199. \int \frac{dx}{vX\sqrt{X}} = \frac{1}{k} \left(\frac{b'}{\sqrt{X}} - \frac{1}{2} \beta \int \frac{dx}{X\sqrt{X}} + b'^2 \int \frac{dx}{v\sqrt{X}} \right).$$

$$200. \int \frac{v dx}{X\sqrt{X}} = -\frac{2(2k + \beta v)}{b'q\sqrt{X}}.$$

$$201. \int \frac{v dx}{\sqrt{X}} = \frac{b'\sqrt{X}}{c} - \frac{\beta}{2c} \int \frac{dx}{\sqrt{X}}.$$

$$202. \int v\sqrt{X} dx = \frac{b'X\sqrt{X}}{3c} - \frac{\beta}{2c} \int \sqrt{X} dx.$$

$$203. \int \frac{v dx}{X^n\sqrt{X}} = -\frac{b'\sqrt{X}}{(2n-1)cX^n} - \frac{\beta}{2c} \int \frac{dx}{X^n\sqrt{X}}.$$

$$204. \int \frac{vX^n dx}{\sqrt{X}} = \frac{b'X^n\sqrt{X}}{(2n+1)c} - \frac{\beta}{2c} \int \frac{X^n dx}{\sqrt{X}}.$$

$$205. \int \frac{dx}{v^m\sqrt{X}} = -\frac{b'\sqrt{X}}{(m-1)kv^{m-1}} - \frac{(2m-3)\beta}{2(m-1)k} \int \frac{dx}{v^{m-1}\sqrt{X}} \\ - \frac{(m-2)c}{(m-1)k} \int \frac{dx}{v^{m-2}\sqrt{X}}, \text{ if } k \neq 0.$$

$$206. \int \frac{dx}{v^m\sqrt{X}} = -\frac{2b'\sqrt{X}}{(2m-1)\beta v^m} \\ - \frac{2(m-1)c}{(2m-1)\beta} \int \frac{dx}{v^{m-1}\sqrt{X}}, \text{ if } k = 0.$$

$$207. \int \frac{\sqrt{X} dx}{v^m} = -\frac{b'X\sqrt{X}}{(m-1)kv^{m-1}} - \frac{(2m-5)\beta}{2(m-1)k} \int \frac{\sqrt{X} dx}{v^{m-1}} \\ - \frac{(m-4)c}{(m-1)k} \int \frac{\sqrt{X} dx}{v^{m-2}} \\ = \frac{1}{(m-1)b'^2} \left(-\frac{b'\sqrt{X}}{v^{m-1}} + \frac{1}{2} \beta \int \frac{dx}{v^{m-1}\sqrt{X}} + c \int \frac{dx}{v^{m-2}\sqrt{X}} \right) \\ = \frac{1}{(m-2)b'^2} \left(-\frac{b'\sqrt{X}}{v^{m-1}} - k \int \frac{dx}{v^m\sqrt{X}} - \frac{1}{2} \beta \int \frac{dx}{v^{m-1}\sqrt{X}} \right).$$

$$208. \int v^m \sqrt{X} dx = \frac{1}{(m+2)c} \left(b' v^{m-1} X \sqrt{X} - (m + \frac{1}{2}) \beta \int v^{m-1} \sqrt{X} dx - (m-1) k \int v^{m-2} \sqrt{X} dx \right).$$

$$209. \int \frac{dx}{v^m X^n \sqrt{X}} = -\frac{1}{(m-1)k} \left(\frac{b' \sqrt{X}}{v^{m-1} X^n} + (m+n-\frac{3}{2}) \beta \int \frac{dx}{v^{m-1} X^n \sqrt{X}} + (m+2n-2)c \int \frac{dx}{v^{m-2} X^n \sqrt{X}} \right), \text{ if } k \neq 0.$$

$$210. \int \frac{dx}{v^m X^n \sqrt{X}} = \frac{-2}{(2m+2n-1)\beta} \left(\frac{b' \sqrt{X}}{v^m X^n} + (m+2n-1)c \int \frac{dx}{v^{m-1} X^n \sqrt{X}} \right), \text{ if } k = 0.$$

$$211. \int \frac{X^n dx}{v^m \sqrt{X}} = -\frac{1}{(m-1)k} \left(\frac{b' X^n \sqrt{X}}{v^{m-1}} + (m-n-\frac{3}{2}) \beta \int \frac{X^n dx}{v^{m-1} \sqrt{X}} + (m-2n-2)c \int \frac{X^n dx}{v^{m-2} \sqrt{X}} \right) \\ = -\frac{1}{(m-2n)b'^2} \left(\frac{b' X^{n-1} \sqrt{X}}{v^{m-1}} + (2n-1)k \int \frac{X^{n-1} dx}{v^m \sqrt{X}} + (n-\frac{1}{2}) \beta \int \frac{X^{n-1} dx}{v^{m-1} \sqrt{X}} \right) \\ = \frac{1}{(m-1)b'^2} \left(-\frac{b' X^{n-1} \sqrt{X}}{v^{m-1}} + (n-\frac{1}{2}) \beta \int \frac{X^{n-1} dx}{v^{m-1} \sqrt{X}} + (2n-1)c \int \frac{X^{n-1} dx}{v^{m-2} \sqrt{X}} \right).$$

$$212. \int \frac{v^m X^n dx}{\sqrt{X}} = \frac{1}{(m+2n)c} \left(b' v^{m-1} X^n \sqrt{X} \right. \\ \left. - (m+n-\frac{1}{2})\beta \int \frac{v^{m-1} X^n dx}{\sqrt{X}} - (m-1)k \int \frac{v^{m-2} X^n dx}{\sqrt{X}} \right)$$

$$213. \int \frac{v^m dx}{X^n \sqrt{X}} = \frac{1}{(m-2n)c} \left(\frac{b' v^{m-1} \sqrt{X}}{X^n} \right. \\ \left. - (m-n-\frac{1}{2})\beta \int \frac{v^{m-1} dx}{X^n \sqrt{X}} - (m-1)k \int \frac{v^{m-2} dx}{X^n \sqrt{X}} \right).$$

$$\frac{1}{(x+a)(x+b)\sqrt{X}} = \frac{1}{(b-a)(x+a)\sqrt{X}} + \frac{1}{(a-b)(x+b)\sqrt{X}}$$

$$\frac{1}{\sqrt{a+bx+cx^2 \pm \sqrt{a'+b'x+c'x^2}}} \\ = \frac{\sqrt{a+bx+cx^2} \mp \sqrt{a'+b'x+c'x^2}}{a-a'+(b-b')x+(c-c')x^2}.$$

$$\frac{\sqrt{X}}{(x+a)(x+b)} = \frac{\sqrt{X}}{(b-a)(x+a)} + \frac{\sqrt{X}}{(a-b)(x+b)}.$$

$$\frac{(x+a)\sqrt{X}}{x+b} = \sqrt{X} + \frac{(a-b)\sqrt{X}}{x+b}.$$

$$\int \sqrt{\frac{ax^2+b}{a'x^2+b'}} dx \text{ is an elliptic integral.}$$

$$\int \frac{x\sqrt{a+bx^2}}{\sqrt{a'+b'x^2}} dx = \frac{1}{b'\sqrt{b'}} \int \sqrt{ab'-a'b+by^2} \cdot dy,$$

where

$$y^2 = a' + b'x^2.$$

IV. MISCELLANEOUS ALGEBRAIC EXPRESSIONS.

$$214. \int \sqrt{2ax - x^2} \cdot dx = \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x-a}{a}.$$

$$215. \int \frac{dx}{\sqrt{2ax - x^2}} = \text{versin}^{-1} \frac{x}{a} = \cos^{-1} \left(1 - \frac{x}{a} \right) \\ = 2 \sin^{-1} \sqrt{\frac{x}{2a}}.$$

$$216. \int \frac{x^n dx}{\sqrt{2ax - x^2}} = - \frac{x^{n-1} \sqrt{2ax - x^2}}{n} \\ - \frac{a(1-2n)}{n} \int \frac{x^{n-1} dx}{\sqrt{2ax - x^2}}.$$

$$217. \int \frac{dx}{x^n \sqrt{2ax - x^2}} = \frac{\sqrt{2ax - x^2}}{a(1-2n)x^n} \\ + \frac{n-1}{(2n-1)a} \int \frac{dx}{x^{n-1} \sqrt{2ax - x^2}}.$$

$$218. \int x^n \sqrt{2ax - x^2} \cdot dx = - \frac{x^{n-1} \sqrt{(2ax - x^2)^3}}{n+2} \\ + \frac{(2n+1)a}{n+2} \int x^{n-1} \sqrt{2ax - x^2} \cdot dx.$$

$$219. \int \frac{\sqrt{2ax - x^2} \cdot dx}{x^n} = \frac{\sqrt{(2ax - x^2)^2}}{(3-2n)ax^n} \\ + \frac{n-3}{(2n-3)a} \int \frac{\sqrt{2ax - x^2} \cdot dx}{x^{n-1}}.$$

$$220. \int \frac{dx}{x \sqrt{x^n - a^2}} = \frac{2}{an} \sec^{-1} \left(\frac{x^{\frac{n}{2}}}{a} \right).$$

$$221. \int \frac{dx}{x\sqrt{x^n+a^2}} = \frac{1}{an} \log \frac{\sqrt{a^2+x^n}-a}{\sqrt{a^2+x^n}+a}.$$

$$222. \int \frac{x^{\frac{1}{2}} dx}{\sqrt{a^3-x^3}} = \frac{2}{3} \sin^{-1} \left(\frac{x}{a} \right)^{\frac{3}{2}}.$$

$$223. \int \frac{dx}{(a+bx^2)\sqrt{x}} = \frac{1}{b\delta^3\sqrt{2}} \left\{ \log \left(\frac{x+\delta^2+\sqrt{2\delta^2x}}{\sqrt{a+bx^2}} \right) + \tan^{-1} \left(1 + \frac{\sqrt{2x}}{\delta} \right) - \tan^{-1} \left(1 - \frac{\sqrt{2x}}{\delta} \right) \right\}, \text{ where } b\delta^4 = a.$$

$$224. \int \frac{\sqrt{x} \cdot dx}{a+bx^2} = \frac{1}{b\delta\sqrt{2}} \left\{ \tan^{-1} \left(1 + \frac{\sqrt{2x}}{\delta} \right) - \tan^{-1} \left(1 - \frac{\sqrt{2x}}{\delta} \right) - \log \left(\frac{x+\delta^2+\sqrt{2\delta^2x}}{\sqrt{a+bx^2}} \right) \right\}, \text{ where } b\delta^4 = a.$$

$$225. \int \frac{x^{\frac{3}{2}} \cdot dx}{a+bx^2} = \frac{2\sqrt{x}}{b} - \frac{a}{b} \int \frac{dx}{(a+bx^2)\sqrt{x}}.$$

$$226. \int \frac{dx}{(a+bx^2)^2\sqrt{x}} = \frac{\sqrt{x}}{2a(a+bx^2)} + \frac{3}{4a} \int \frac{dx}{(a+bx^2)\sqrt{x}}.$$

$$227. \int \frac{\sqrt{x} \cdot dx}{(a+bx^2)^2} = \frac{x^{\frac{3}{2}}}{2a(a+bx^2)} + \frac{1}{4a} \int \frac{\sqrt{x} \cdot dx}{(a+bx^2)}.$$

If a_1, a_2, a_3 , etc., are the roots of the equation

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0,$$

the integrand in the expression

$$\int \frac{(q_0x^m + q_1x^{m-1} + \dots + q_n)dx}{(p_0x^n + p_1x^{n-1} + \dots + p_n)\sqrt{a+bx+cx^2}},$$

where $m < n$, may be expressed as the sum of a number of partial fractions of the form $\frac{A}{(x - a_k)^r \sqrt{a + bx + cx^2}}$, and these can be integrated by the aid of equations given above. Thus,

$$\begin{aligned}
 228. \int \frac{(px + q) dx}{(x - a')(x - b') \sqrt{a + bx + cx^2}} \\
 = \frac{q + a'p}{a' - b'} \int \frac{dx}{(x - a') \sqrt{a + bx + cx^2}} \\
 - \frac{q + b'p}{a' - b'} \int \frac{dx}{(x - b') \sqrt{a + bx + cx^2}}.
 \end{aligned}$$

$$\begin{aligned}
 229. \int \frac{dx}{(a' + c'x^2) \sqrt{a + cx^2}} \\
 = \frac{1}{\sqrt{a'(ac' - a'e)}} \tan^{-1} x \sqrt{\frac{(ac' - a'e)}{a'(a + cx^2)}} \\
 = \frac{1}{2\sqrt{a'(a'e - ac')}} \log \frac{\sqrt{a'(a + cx^2)} + x \sqrt{a'e - ac'}}{\sqrt{a'(a + cx^2)} - x \sqrt{a'e - ac'}}
 \end{aligned}$$

$$\begin{aligned}
 230. \int \frac{x dx}{(a' + c'x^2) \sqrt{a + cx^2}} \\
 = \frac{1}{\sqrt{c'(a'e - ac')}} \tan^{-1} \sqrt{\frac{c'(a + cx^2)}{a'e - ac'}} \\
 = \frac{1}{2\sqrt{c'(ac' - a'e)}} \log \frac{\sqrt{c'(a + cx^2)} - \sqrt{ac' - a'e}}{\sqrt{c'(a + cx^2)} + \sqrt{ac' - a'e}}
 \end{aligned}$$

$$\begin{aligned}
 231. \int f \left\{ x, \sqrt[n]{\frac{a + bx}{a' + b'x}} \right\} dx \\
 = n(a'b - ab') \int f \left(\frac{a - a'z^n}{b'z^n - b}, z \right) \cdot \frac{z^{n-1} dz}{(b'z^n - b)^2},
 \end{aligned}$$

where $z^n(a' + b'x) = a + bx$.

$$\begin{aligned}
 232. \int f(x, \sqrt[n]{c + \sqrt[m]{a + bx}}) dx \\
 = \frac{mn}{b} \int f\left\{ \frac{(z^n - c)^m - a}{b}, z \right\} (z^n - c)^{m-1} z^{n-1} dz,
 \end{aligned}$$

where $z^n = c + \sqrt[n]{a + bx}$.

$$\begin{aligned}
 233. \int f\left\{ x, \left[\frac{a + bx}{a' + b'x} \right]^{\frac{m}{n}}, \left[\frac{a + bx}{a' + b'x} \right]^{\frac{p}{q}}, \dots \right\} dx \\
 = s(a'b - ab') \int f\left\{ \frac{a'y^s - a}{b - b'y^s}, y^{\frac{ms}{n}}, y^{\frac{ps}{q}}, \dots \right\} \frac{y^{s-1} dy}{(b - b'y^s)^2},
 \end{aligned}$$

where $y^s(a' + b'x) = a + bx$ and s is the least common multiple of n, q , etc.

$$\begin{aligned}
 234. \int f(x, \sqrt{a + bx + x^2}) dx \\
 = 2 \int f\left(\frac{2\sqrt{a} \cdot z - b}{1 - z^2}, \frac{z^2\sqrt{a} - bz + \sqrt{a}}{1 - z^2} \right) \cdot \frac{(z^2\sqrt{a} - bz + \sqrt{a}) dz}{(1 - z^2)^2},
 \end{aligned}$$

where $xz + \sqrt{a} = \sqrt{a + bx + x^2}$.

$$\begin{aligned}
 235. \int f(x, \sqrt{a + bx + x^2}) dx \\
 = \int f\left(\frac{u^2 - a}{b - 2u}, \frac{u^2 - bu + a}{2u - b} \right) \frac{2(bu - a - u^2) du}{(b - 2u)^2},
 \end{aligned}$$

where $u = \sqrt{a + bx + x^2} - x$.

$$\int \frac{dx}{x^4 + a^4} = \frac{1}{4a^3\sqrt{2}} \left\{ \log \left(\frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2} \right) + 2 \tan^{-1} \left(\frac{ax\sqrt{2}}{a^2 - x^2} \right) \right\}.$$

$$\int \frac{dx}{x^4 - a^4} = \frac{1}{4a^3} \left\{ \log \left(\frac{x - a}{x + a} \right) - 2 \tan^{-1} \left(\frac{x}{a} \right) \right\}.$$

V. TRANSCENDENTAL FUNCTIONS.

$$236. \int \sin x \cdot f(\cos x) dx = - \int f(\cos x) d \cos x.$$

$$237. \int \cos x \cdot f(\sin x) dx = \int f(\sin x) d \sin x.$$

$$238. \int \sin x \cdot f(\sin x, \cos x) dx = - \int f(\sqrt{1-z^2}, z) dz,$$

where $z = \cos x$.

$$239. \int \frac{dx}{a + b \cos x} = \frac{1}{c(b-a)} \left\{ \int \frac{dz}{z+c} - \int \frac{dz}{z-c} \right\},$$

where $z = \tan \frac{1}{2} x$, and $c^2 = (b+a)/(b-a)$. [See 651.]

$$240. \int \frac{dx}{a \pm b \sin x} = \int \frac{2 dz}{a \pm 2bz + az^2}, \text{ where } z = \tan \frac{1}{2} x.$$

$$241. \int f(\sin x) dx = - \int f\left(\cos\left(\frac{\pi}{2} - x\right)\right) d\left(\frac{\pi}{2} - x\right).$$

$$242. \int f(\tan x) dx = - \int f \operatorname{ctn}\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right).$$

$$243. \int f(\sec x) dx = - \int f \operatorname{csc}\left(\frac{\pi}{2} - x\right) d\left(\frac{\pi}{2} - x\right).$$

$$244. \int \frac{\sin x \cdot f(\sin^2 x) dx}{\sqrt{1-k^2 \sin^2 x}} = \int \frac{f(z) dz}{2\sqrt{(1-z)(1-k^2 z)}},$$

where $z = \sin^2 x$.

$$245. \int \frac{\cos x \cdot f(\cos^2 x) dx}{\sqrt{1-k^2 \sin^2 x}} = \int \frac{f(1-z) dz}{2\sqrt{z(1-k^2 z)}}, \text{ where } z = \sin^2 x.$$

$$246. \int \frac{\tan x \cdot f(\tan^2 x) dx}{\sqrt{1 - k^2 \sin^2 x}} = \int f\left(\frac{z}{1 - z}\right) \frac{dz}{2(1 - z)\sqrt{1 - k^2 z}},$$

where $z = \sin^2 x$.

$$247. \int f(ax + b) dx = \frac{1}{a} \int f(ax + b) d(ax + b).$$

$$248. \int \sec^{n+2} x \cdot f(\tan x) dx = \int (1 + z^2)^{\frac{n}{2}} f(z) dz; \quad z = \tan x.$$

$$249. \int f(\sin x, \cos x) dx \\ = - \int f\left(\cos\left(\frac{\pi}{2} - x\right), \sin\left(\frac{\pi}{2} - x\right)\right) d\left(\frac{\pi}{2} - x\right).$$

$$250. \int f(x) \cdot \sin^{-1} x \cdot dx = \sin^{-1} x \cdot \phi(x) - \int \frac{\phi(x) dx}{\sqrt{1 - x^2}}, \quad dx,$$

where $\phi(x) = \int f(x) dx$.

$$251. \int f(x) \cdot \cos^{-1} x \cdot dx = \cos^{-1} x \cdot \phi(x) + \int \frac{\phi(x) dx}{\sqrt{1 - x^2}}.$$

$$252. \int f(x) \cdot \tan^{-1} x \cdot dx = \tan^{-1} x \cdot \phi(x) - \int \frac{\phi(x) dx}{1 + x^2}.$$

$$253. \int f(x) \cdot \text{ctn}^{-1} x \cdot dx = \text{ctn}^{-1} x \cdot \phi(x) + \int \frac{\phi(x) dx}{1 + x^2}.$$

$$254. \int f(x, \cos x) dx = - \int f\left(\frac{\pi}{2} - z, \sin z\right) dz,$$

where $z = \frac{\pi}{2} - x$.

$$255. \int \frac{\sin x \cdot f(\cos x) dx}{a + b \cos x} = - \frac{1}{b} \int f\left(\frac{z - a}{b}\right) \frac{dz}{z},$$

where $z = a + b \cos x$.

$$256. \int f(x, \log x) dx = \int f(e^z, z) e^z dz, \text{ where } z = \log x.$$

$$257. \int \frac{f(\log x) dx}{x} = \int f(z) dz, \text{ where } z = \log x.$$

$$258. \int x^m f(\log x) dx = \int e^{(m+1)z} f(z) dz.$$

$$259. \int f(\sin x, \cos x, \tan x, \cot x, \sec x, \csc x) dx \\ = \int f\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}, \frac{2z}{1-z^2}, \frac{1-z^2}{2z}, \frac{1+z^2}{1-z^2}, \frac{1+z^2}{2z}\right)$$

$$\frac{2 dz}{1+z^2}, \text{ where } z = \tan \frac{x}{2};$$

$$= \int f\left(z, \sqrt{1-z^2}, \frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, \frac{1}{\sqrt{1-z^2}}, \frac{1}{z}\right)$$

$$\frac{dz}{\sqrt{1-z^2}}, \text{ where } z = \sin x;$$

$$= \int f\left(\frac{z}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+z^2}}, z, \frac{1}{z}, \sqrt{1+z^2}, \frac{\sqrt{1+z^2}}{z}\right)$$

$$\frac{dz}{1+z^2}, \text{ where } z = \tan x;$$

$$= \int f\left(\sqrt{z}, \sqrt{1-z}, \sqrt{\frac{z}{1-z}}, \sqrt{\frac{1-z}{z}}, \frac{1}{\sqrt{1-z}}, \frac{1}{\sqrt{z}}\right)$$

$$\frac{dz}{2\sqrt{z(1-z)}}, \text{ where } z = \sin^2 x;$$

$$= \int f\left(\sqrt{\frac{z}{1+z}}, \frac{1}{\sqrt{1+z}}, \sqrt{z}, \frac{1}{\sqrt{z}}, \sqrt{1+z}, \sqrt{\frac{1+z}{z}}\right)$$

$$\frac{dz}{2\sqrt{z(1+z)}}, \text{ where } z = \tan^2 x.$$

$$260. \int \sin x \, dx = -\cos x. \quad [\text{See 247.}]$$

$$261. \int \sin^2 x \, dx = -\frac{1}{2} \cos x \sin x + \frac{1}{2} x = \frac{1}{2} x - \frac{1}{4} \sin 2x.$$

$$262. \int \sin^3 x \, dx = -\frac{1}{3} \cos x (\sin^2 x + 2).$$

$$263. \int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

$$264. \int \cos x \, dx = \sin x. \quad [\text{See 247.}]$$

$$265. \int \cos^2 x \, dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} x = \frac{1}{2} x + \frac{1}{4} \sin 2x.$$

$$266. \int \cos^3 x \, dx = \frac{1}{3} \sin x (\cos^2 x + 2).$$

$$267. \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

$$268. \int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x.$$

$$269. \int \sin^2 x \cos^2 x \, dx = -\frac{1}{8} \left(\frac{1}{4} \sin 4x - x \right).$$

$$270. \int \sin x \cos^m x \, dx = -\frac{\cos^{m+1} x}{m+1}.$$

$$271. \int \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1}.$$

$$272. \int \cos^m x \sin^n x \, dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} \\ + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x \, dx.$$

$$273. \int \cos^m x \sin^n x \, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} \\ + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx.$$

$$\begin{aligned}
 274. \int \frac{\sin^n x \, dx}{\cos^m x} &= \frac{1}{n-m} \left(-\frac{\sin^{n-1} x}{\cos^{m-1} x} + (n-1) \int \frac{\sin^{n-2} x \, dx}{\cos^m x} \right) \\
 &= \frac{1}{m-1} \left(\frac{\sin^{n+1} x}{\cos^{m-1} x} - (n-m+2) \int \frac{\sin^n x \, dx}{\cos^{m-2} x} \right) \\
 &= \frac{1}{m-1} \left(\frac{\sin^{n-1} x}{\cos^{m-1} x} - (n-1) \int \frac{\sin^{n-2} x \, dx}{\cos^{m-2} x} \right).
 \end{aligned}$$

$$\begin{aligned}
 275. \int \frac{\cos^m x \, dx}{\sin^n x} &= -\frac{\cos^{m+1} x}{(n-1) \sin^{n-1} x} - \frac{m-n+2}{n-1} \int \frac{\cos^m x \, dx}{\sin^{n-2} x} \\
 &= \frac{\cos^{m-1} x}{(m-n) \sin^{n-1} x} + \frac{m-1}{m-n} \int \frac{\cos^{m-2} x \, dx}{\sin^n x} \\
 &= -\frac{1}{n-1} \frac{\cos^{m-1} x}{\sin^{n-1} x} - \frac{m-1}{n-1} \int \frac{\cos^{m-2} x \, dx}{\sin^{n-2} x}.
 \end{aligned}$$

$$276. \int \frac{\sin^m x \, dx}{\cos^n x} = - \int \frac{\cos^m \left(\frac{\pi}{2} - x \right) d \left(\frac{\pi}{2} - x \right)}{\sin^n \left(\frac{\pi}{2} - x \right)}.$$

$$277. \int \frac{dx}{\sin x \cos x} = \log \tan x.$$

$$278. \int \frac{dx}{\cos x \sin^2 x} = \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) - \csc x.$$

$$\begin{aligned}
 279. \int \frac{dx}{\sin^m x \cos^n x} \\
 &= \frac{1}{n-1} \cdot \frac{1}{\sin^{m-1} x \cdot \cos^{n-1} x} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m x \cdot \cos^{n-2} x} \\
 &= -\frac{1}{m-1} \cdot \frac{1}{\sin^{m-1} x \cdot \cos^{n-1} x} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^{m-2} x \cdot \cos^n x}.
 \end{aligned}$$

$$280. \int \frac{dx}{\sin^m x} = -\frac{1}{m-1} \cdot \frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1} \int \frac{dx}{\sin^{m-2} x}.$$

$$281. \int \frac{dx}{\cos^n x} = \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.$$

$$282. \int \tan x \, dx = -\log \cos x. \quad [\text{See 247.}]$$

$$283. \int \tan^2 x \, dx = \tan x - x.$$

$$284. \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx.$$

$$285. \int \operatorname{ctn} x \, dx = \log \sin x. \quad [\text{See 247.}]$$

$$286. \int \operatorname{ctn}^2 x \, dx = -\operatorname{ctn} x - x.$$

$$287. \int \operatorname{ctn}^n x \, dx = -\frac{\operatorname{ctn}^{n-1} x}{n-1} - \int \operatorname{ctn}^{n-2} x \, dx.$$

$$288. \int \sec x \, dx = \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) = \frac{1}{2} \log \frac{1 + \sin x}{1 - \sin x}.$$

$$289. \int \sec^2 x \, dx = \tan x.$$

$$\begin{aligned} 290. \int \sec^n x \, dx &= \int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1) \cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x} \\ &= \frac{\sin x}{(n-1) \cos^{n-1} x} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx. \end{aligned}$$

$$291. \int \csc x \, dx = \log \tan \frac{1}{2} x.$$

$$292. \int \csc^2 x \, dx = -\operatorname{ctn} x.$$

$$\begin{aligned}
 293. \int \csc^n x \, dx &= \int \frac{dx}{\sin^n x} \\
 &= -\frac{\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2}x} \\
 &= -\frac{\cos x}{(n-1)\sin^{n-1}x} + \frac{n-2}{n-1} \int \csc^{n-2} x \, dx.
 \end{aligned}$$

$$294. \int \frac{dx}{1 + \sin x} = -\tan\left(\frac{1}{4}\pi - \frac{1}{2}x\right). \quad [\text{See 241.}]$$

$$295. \int \frac{dx}{1 - \sin x} = \operatorname{ctn}\left(\frac{1}{4}\pi - \frac{1}{2}x\right) = \tan\left(\frac{1}{4}\pi + \frac{1}{2}x\right).$$

$$296. \int \frac{dx}{1 + \cos x} = \tan \frac{1}{2}x, \text{ or } \csc x - \operatorname{ctn} x.$$

$$297. \int \frac{dx}{1 - \cos x} = -\operatorname{ctn} \frac{1}{2}x, \text{ or } -\operatorname{ctn} x - \csc x.$$

$$298. \int \frac{dx}{a \pm b \sin x} = \frac{2 \sec \theta}{a} \cdot \tan^{-1}(\sec \theta \cdot \tan \frac{1}{2}x \pm \tan \theta),$$

if $a > b$, and $b = a \sin \theta$.

$$299. \int \frac{dx}{a \pm b \sin x} = \frac{\pm \sec a}{b} \log \frac{\sin \frac{1}{2}(a \pm x)}{\cos \frac{1}{2}(x \mp a)},$$

if $b > a$, and $a = b \sin a$. [See 241.]

$$300. \int \frac{dx}{a + b \cos x} = \frac{-1}{\sqrt{a^2 - b^2}} \cdot \sin^{-1} \left[\frac{b + a \cos x}{a + b \cos x} \right],$$

$$\text{or } \frac{1}{\sqrt{a^2 - b^2}} \sin^{-1} \left[\frac{\sqrt{a^2 - b^2} \cdot \sin x}{a + b \cos x} \right],$$

$$\text{or } \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\sqrt{\frac{a-b}{a+b}} \tan \frac{1}{2}x \right],$$

$$\text{or } \frac{1}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\frac{\sqrt{a^2 - b^2} \cdot \sin x}{b + a \cos x} \right],$$

[For the interpretation of these
antitrigonometric functions in any
doubtful case, see 234.]

$$\text{or } \frac{1}{\sqrt{b^2 - a^2}} \log \left[\frac{b + a \cos x + \sqrt{b^2 - a^2} \cdot \sin x}{a + b \cos x} \right],$$

$$\text{or } \frac{1}{\sqrt{b^2 - a^2}} \log \left[\frac{\sqrt{b + a} + \sqrt{b - a} \cdot \tan \frac{1}{2} x}{\sqrt{b + a} - \sqrt{b - a} \cdot \tan \frac{1}{2} x} \right],$$

$$\text{or } \frac{1}{\sqrt{b^2 - a^2}} \tanh^{-1} \left[\frac{\sqrt{b^2 - a^2} \cdot \sin x}{b + a \cos x} \right].$$

$$301. \int \frac{dx}{a + b \tan x} = \frac{1}{a^2 + b^2} [b \log (a \cos x + b \sin x) + ax].$$

$$302. \int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log \tan \left(\frac{1}{2} x + \frac{1}{8} \pi \right).$$

$$303. \int \frac{\sin x \, dx}{a + b \cos x} = - \int \frac{\cos (\frac{1}{2} \pi - x) \, d(\frac{1}{2} \pi - x)}{a + b \sin (\frac{1}{2} \pi - x)} \\ = - \frac{1}{b} \log (a + b \cos x).$$

$$304. \int \frac{(a' + b' \cos x) \, dx}{a + b \cos x} = \frac{b'x}{b} + \frac{a'b - ab'}{b} \int \frac{dx}{a + b \cos x}.$$

$$305. \int \frac{(a' + b' \cos x) \, dx}{(a + b \cos x)^2} = \frac{ab' - a'b}{a^2 - b^2} \frac{\sin x}{a + b \cos x} \\ + \frac{aa' - bb'}{a^2 - b^2} \int \frac{dx}{a + b \cos x}. \quad [\text{See 241.}]$$

$$306. \int \frac{(a' + b' \cos x) \, dx}{(a + b \cos x)^n} = \frac{1}{(n-1)(a^2 - b^2)} \left[\frac{(ab' - a'b) \sin x}{(a + b \cos x)^{n-1}} \right. \\ \left. + \int \frac{[(aa' - bb')(n-1) + (n-2)(ab' - a'b) \cos x] \, dx}{(a + b \cos x)^{n-1}} \right].$$

$$307. \int \frac{(a' + b' \cos x) dx}{(1 + \cos x)^n} = \frac{(a' - b') \tan \frac{1}{2} x}{(2n - 1)(1 + \cos x)^{n-1}} \\ + \frac{n(a' + b') - a'}{2n - 1} \int \frac{dx}{(1 + \cos x)^{n-1}}.$$

$$308. \int \frac{dx}{(a + b \cos x)^n} = \frac{1}{(n - 1)(a^2 - b^2)} \left[\frac{-b \sin x}{(a + b \cos x)^{n-1}} \right. \\ \left. + (2n - 3)a \int \frac{dx}{(a + b \cos x)^{n-1}} - (n - 2) \int \frac{dx}{(a + b \cos x)^{n-2}} \right].$$

$$309. \int \frac{dx}{(1 + \cos x)^n} = \frac{\tan \frac{1}{2} x}{(2n - 1)(1 + \cos x)^{n-1}} \\ + \frac{n - 1}{2n - 1} \int \frac{dx}{(1 + \cos x)^{n-1}}. \quad [\text{See 241.}]$$

$$310. \int \frac{(a' + b' \cos x) dx}{\sin x (a + b \cos x)} = \frac{a'b - ab'}{a^2 - b^2} \log(a + b \cos x) \\ + \frac{a' + b'}{a + b} \log \sin \frac{1}{2} x - \frac{a' - b'}{a - b} \log \cos \frac{1}{2} x.$$

$$311. \int \frac{(a' + b' \cos x) dx}{\cos x (a + b \cos x)} = \frac{a'}{a} \log \tan \frac{1}{2} (\frac{1}{2} \pi + x) \\ + \frac{(ab' - a'b)}{a} \int \frac{dx}{a + b \cos x}.$$

$$312. \int \frac{(a' + b' \cos x) dx}{\sin x (1 \pm \cos x)} = \pm \frac{\frac{1}{2}(a' \mp b')}{1 \pm \cos x} + \frac{1}{2}(a' \pm b') \log \tan \frac{1}{2} x$$

$$313. \int \frac{dx}{(1 - \cos x)^n} = \frac{-\operatorname{ctn} \frac{1}{2} x}{(2n - 1)(1 - \cos x)^{n-1}} \\ + \frac{n - 1}{2n - 1} \int \frac{dx}{(1 - \cos x)^{n-1}}. \quad [\text{See 241.}]$$

$$\begin{aligned}
 314. \quad \int \frac{dx}{a^2 - b^2 \cos^2 x} &= \int \frac{dx}{(a^2 - b^2) + b^2 \sin^2 x} \\
 &= \frac{1}{2ab \sin a} \log \frac{\sin(a-x)}{\sin(a+x)},
 \end{aligned}$$

or $\frac{1}{a^2 \sin \beta} \tan^{-1} \left(\frac{\tan x}{\sin \beta} \right)$, where $\cos a = \frac{1}{\cos \beta} = \frac{a}{b}$.

$$315. \quad \int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{ab} \tan^{-1} \left(\frac{b \tan x}{a} \right).$$

$$316. \quad \int \frac{\sin^2 x dx}{a + b \cos^2 x} = \frac{\sqrt{a+b}}{b\sqrt{a}} \tan^{-1} \left(\tan x \cdot \sqrt{\frac{a}{a+b}} \right) - \frac{x}{b}.$$

$$317. \quad \int \frac{\sin x \cos x dx}{a \cos^2 x + b \sin^2 x} = \frac{1}{2(b-a)} \log(a \cos^2 x + b \sin^2 x).$$

$$318. \quad \int \frac{dx}{(a + b \cos x + c \sin x)^n} = \int \frac{d(x-a)}{[a + r \cos(x-a)]^n},$$

where $b = r \cos a$ and $c = r \sin a$.

$$\begin{aligned}
 319. \quad \int \frac{dx}{a + b \cos x + c \sin x} & \quad [\text{See page 61.}] \\
 &= \frac{-1}{\sqrt{a^2 - b^2 - c^2}} \cdot \sin^{-1} \left[\frac{b^2 + c^2 + a(b \cos x + c \sin x)}{\sqrt{(b^2 + c^2)}(a + b \cos x + c \sin x)} \right] \\
 &= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \cdot \log \\
 &\quad \left[\frac{b^2 + c^2 + a(b \cos x + c \sin x) + \sqrt{b^2 + c^2 - a^2}(b \sin x - c \cos x)}{\sqrt{(b^2 + c^2)}(a + b \cos x + c \sin x)} \right] \\
 &= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \cdot \log \frac{\sqrt{b^2 + c^2 - a^2} - c + (b-a) \tan \frac{1}{2} x}{\sqrt{b^2 + c^2 - a^2} + c - (b-a) \tan \frac{1}{2} x} \\
 &= \frac{2}{\sqrt{a^2 - b^2 - c^2}} \tan^{-1} \left[\frac{(a-b) \tan \frac{1}{2} x + c}{\sqrt{a^2 - b^2 - c^2}} \right].
 \end{aligned}$$

$$320. \int \frac{dx}{a(1 + \cos x) + c \sin x} = \frac{1}{c} \log(a + c \tan \tfrac{1}{2} x)$$

$$321. \int \frac{dx}{(a[1 + \cos x] + c \sin x)^2} \\ = \frac{1}{c^3} \left[\frac{c(a \sin x - c \cos x)}{a(1 + \cos x) + c \sin x} - a \log(a + c \tan \tfrac{1}{2} x) \right].$$

$$322. \int \frac{(x + \sin x) dx}{1 + \cos x} = x \tan \tfrac{1}{2} x.$$

$$323. \int \cos x \sqrt{1 - k^2 \sin^2 x} dx \\ = \tfrac{1}{2} \sin x \sqrt{1 - k^2 \sin^2 x} + \frac{1}{2k} \sin^{-1}(k \sin x).$$

$$324. \int \sin x \sqrt{1 - k^2 \sin^2 x} dx \\ = -\tfrac{1}{2} \cos x \sqrt{1 - k^2 \sin^2 x} - \frac{1 - k^2}{2k} \log(k \cos x + \sqrt{1 - k^2 \sin^2 x}).$$

$$325. \int \sin x (1 - k^2 \sin^2 x)^{\frac{3}{2}} dx = -\tfrac{1}{4} \cos x (1 - k^2 \sin^2 x)^{\frac{3}{2}} \\ + \tfrac{3}{4} (1 - k^2) \int \sin x \sqrt{1 - k^2 \sin^2 x} dx.$$

$$326. \int \frac{\cos x dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{k} \sin^{-1}(k \sin x), \\ \text{or } \frac{1}{b} \log(b \sin x + \sqrt{1 + b^2 \sin^2 x}), \text{ where } b^2 = -k^2.$$

$$327. \int \frac{\sin x dx}{\sqrt{1 - k^2 \sin^2 x}} = -\frac{1}{k} \log(k \cos x + \sqrt{1 - k^2 \sin^2 x}), \\ \text{or } -\frac{1}{b} \sin^{-1} \frac{b \cos x}{\sqrt{1 + b^2}}, \text{ where } b^2 = -k^2$$

$$328. \int \frac{\tan x dx}{\sqrt{1 - k^2 \sin^2 x}} \\ = \frac{1}{2\sqrt{1 - k^2}} \log \left(\frac{\sqrt{1 - k^2 \sin^2 x} + \sqrt{1 - k^2}}{\sqrt{1 - k^2 \sin^2 x} - \sqrt{1 - k^2}} \right).$$

$$329. \int \frac{x dx}{1 + \sin x} = -x \tan \frac{1}{2} (\frac{1}{2} \pi - x) + 2 \log \cos \frac{1}{2} (\frac{1}{2} \pi - x).$$

$$330. \int \frac{x dx}{1 - \sin x} = x \operatorname{ctn} \frac{1}{2} (\frac{1}{2} \pi - x) + 2 \log \sin \frac{1}{2} (\frac{1}{2} \pi - x).$$

$$331. \int \frac{x dx}{1 + \cos x} = x \tan \frac{1}{2} x + 2 \log \cos \frac{1}{2} x.$$

$$332. \int \frac{x dx}{1 - \cos x} = -x \operatorname{ctn} \frac{1}{2} x + 2 \log \sin \frac{1}{2} x.$$

$$333. \int \frac{\tan x dx}{\sqrt{a + b \tan^2 x}} = \frac{1}{\sqrt{b - a}} \cos^{-1} \left(\frac{\sqrt{b - a}}{\sqrt{b}} \cdot \cos x \right).$$

$$334. \int \frac{dx}{a + b \tan^2 x} = \frac{1}{a - b} \left[x - \sqrt{\frac{b}{a}} \cdot \tan^{-1} \left(\sqrt{\frac{b}{a}} \cdot \tan x \right) \right].$$

$$335. \int \frac{\tan x dx}{a + b \tan x} \\ = \frac{1}{a^2 + b^2} \left\{ bx - a \log(a + b \tan x) + a \log \sec x \right\}.$$

$$336. \int x \sin x dx = \sin x - x \cos x.$$

$$337. \int x^2 \sin x dx = 2x \sin x - (x^2 - 2) \cos x.$$

$$338. \int x^3 \sin x dx = (3x^2 - 6) \sin x - (x^3 - 6x) \cos x.$$

$$339. \int x^m \sin x dx = -x^m \cos x + m \int x^{m-1} \cos x dx.$$

$$340. \int x \cos x dx = \cos x + x \sin x.$$

$$341. \int x^2 \cos x dx = 2x \cos x + (x^2 - 2) \sin x.$$

$$342. \int x^3 \cos x dx = (3x^2 - 6) \cos x + (x^3 - 6x) \sin x.$$

$$343. \int x^m \cos x \, dx = x^m \sin x - m \int x^{m-1} \sin x \, dx.$$

$$344. \int \frac{\sin x}{x^m} \, dx = -\frac{1}{m-1} \cdot \frac{\sin x}{x^{m-1}} + \frac{1}{m-1} \int \frac{\cos x}{x^{m-1}} \, dx.$$

$$345. \int \frac{\cos x}{x^m} \, dx = -\frac{1}{m-1} \cdot \frac{\cos x}{x^{m-1}} - \frac{1}{m-1} \int \frac{\sin x}{x^{m-1}} \, dx.$$

$$346. \int \frac{\sin x}{x} \, dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} \cdots$$

$$347. \int \frac{\cos x}{x} \, dx = \log x - \frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} - \frac{x^6}{6 \cdot 6!} + \frac{x^8}{8 \cdot 8!} \cdots$$

$$348. \int \frac{x \, dx}{\sin x} = x + \frac{x^3}{3 \cdot 3!} + \frac{7x^5}{3 \cdot 5 \cdot 5!} + \frac{31x^7}{3 \cdot 7 \cdot 7!} + \frac{127x^9}{3 \cdot 5 \cdot 9!} + \cdots$$

$$349. \int \frac{x \, dx}{\cos x} = \frac{x^2}{2} + \frac{x^4}{4 \cdot 2!} + \frac{5x^6}{6 \cdot 4!} + \frac{61x^8}{8 \cdot 6!} + \frac{1385x^{10}}{10 \cdot 8!} + \cdots$$

$$350. \int \frac{x \, dx}{\sin^2 x} = -x \cot x + \log \sin x.$$

$$351. \int \frac{x \, dx}{\cos^2 x} = x \tan x + \log \cos x.$$

$$\begin{aligned} 352. \int x^m \sin^n x \, dx \\ = x^{m-1} \sin^{n-1} x (m \sin x - nx \cos x) \\ + n(n-1) \int x^m \sin^{n-2} x \, dx - m(m-1) \int x^{m-2} \sin^n x \, dx. \end{aligned}$$

$$\begin{aligned} 353. \int x^m \cos^n x \, dx \\ = x^{m-1} \cos^{n-1} x (m \cos x + nx \sin x) \\ + n(n-1) \int x^m \cos^{n-2} x \, dx - m(m-1) \int x^{m-2} \cos^n x \, dx. \end{aligned}$$

$$\begin{aligned}
 354. \quad & \int \frac{x^m dx}{\sin^n x} \\
 &= \frac{1}{(n-1)(n-2)} \left[-\frac{x^{m-1}(m \sin x + (n-2)x \cos x)}{\sin^{n-1} x} \right. \\
 & \quad \left. + (n-2)^2 \int \frac{x^m dx}{\sin^{n-2} x} + m(m-1) \int \frac{x^{m-2} dx}{\sin^{n-2} x} \right].
 \end{aligned}$$

$$\begin{aligned}
 355. \quad & \int \frac{x^m dx}{\cos^n x} \\
 &= \frac{1}{(n-1)(n-2)} \left[-\frac{x^{m-1}(m \cos x - (n-2)x \sin x)}{\cos^{n-1} x} \right. \\
 & \quad \left. + (n-2)^2 \int \frac{x^m dx}{\cos^{n-2} x} + m(m-1) \int \frac{x^{m-2} dx}{\cos^{n-2} x} \right].
 \end{aligned}$$

$$\begin{aligned}
 356. \quad & \int \frac{\sin^n x dx}{x^m} \\
 &= \frac{1}{(m-1)(m-2)} \left[-\frac{\sin^{n-1} x ((m-2) \sin x + nx \cos x)}{x^{m-1}} \right. \\
 & \quad \left. - n^2 \int \frac{\sin^n x dx}{x^{m-2}} + n(n-1) \int \frac{\sin^{n-2} x dx}{x^{m-2}} \right].
 \end{aligned}$$

$$\begin{aligned}
 357. \quad & \int \frac{\cos^n x dx}{x^m} \\
 &= \frac{1}{(m-1)(m-2)} \left[\frac{\cos^{n-1} x (nx \cos x - (m-2) \cos x)}{x^{m-1}} \right. \\
 & \quad \left. - n^2 \int \frac{\cos^n x dx}{x^{m-2}} + n(n-1) \int \frac{\cos^{n-2} x dx}{x^{m-2}} \right].
 \end{aligned}$$

$$\begin{aligned}
 358. \quad & \int x^p \sin^m x \cos^n x dx \\
 &= \frac{1}{(m+n)^2} \left[x^{p-1} \sin^m x \cos^{n-1} x (p \cos x + (m+n)x \sin x) \right. \\
 & \quad \left. + (n-1)(m+n) \int x^p \sin^m x \cos^{n-2} x dx \right.
 \end{aligned}$$

$$\begin{aligned}
 & -mp \int x^{p-1} \sin^{m-1} x \cos^{n-1} x dx \\
 & -p(p-1) \int x^{p-2} \sin^m x \cos^n x dx \Big]. \\
 & = \frac{1}{(m+n)^2} \Big[x^{p-1} \sin^{m-1} x \cos^n x (p \sin x - (m+n)x \cos x) \\
 & + (m-1)(m+n) \int x^p \sin^{m-2} x \cos^n x dx \\
 & + np \int x^{p-1} \sin^{m-1} x \cos^{n-1} x dx \\
 & -p(p-1) \int x^{p-2} \sin^m x \cos^n x dx \Big].
 \end{aligned}$$

$$359. \int \sin mx \sin nx dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}.$$

$$360. \int \sin mx \cos nx dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}.$$

$$361. \int \cos mx \cos nx dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}.$$

$$362. \int \sin^2 mx dx = \frac{1}{2m} (mx - \sin mx \cos mx).$$

$$363. \int \cos^2 mx dx = \frac{1}{2m} (mx + \sin mx \cos mx).$$

$$364. \int \sin mx \cos mx dx = -\frac{1}{4m} \cos 2mx.$$

$$\begin{aligned}
 365. \int \sin nx \sin^m x dx &= \frac{1}{m+n} \Big[-\cos nx \sin^m x \\
 &+ m \int \cos(n-1)x \cdot \sin^{m-1} x dx \Big].
 \end{aligned}$$

See page 61.

$$366. \int \sin nx \cos^m x dx = \frac{1}{m+n} \left[-\cos nx \cos^m x + m \int \sin (n-1)x \cdot \cos^{m-1} x dx \right].$$

$$367. \int \cos nx \sin^m x dx = \frac{1}{m+n} \left[\sin nx \sin^m x - m \int \sin (n-1)x \cdot \sin^{m-1} x dx \right].$$

$$368. \int \cos nx \cos^m x dx = \frac{1}{m+n} \left[\sin nx \cos^m x + m \int \cos (n-1)x \cdot \cos^{m-1} x dx \right].$$

$$369. \int \frac{\cos nx dx}{\cos^m x} = 2 \int \frac{\cos (n-1)x dx}{\cos^{m-1} x} - \int \frac{\cos (n-2)x dx}{\cos^m x}.$$

$$370. \int \frac{\cos nx dx}{\sin^m x} = -2 \int \frac{\sin (n-1)x dx}{\sin^{m-1} x} + \int \frac{\cos (n-2)x dx}{\sin^m x}.$$

$$371. \int \frac{\sin nx dx}{\sin^m x} = 2 \int \frac{\cos (n-1)x dx}{\sin^{m-1} x} + \int \frac{\sin (n-2)x dx}{\sin^m x}.$$

$$372. \int \frac{\sin nx dx}{\cos^m x} = 2 \int \frac{\sin (n-1)x dx}{\cos^{m-1} x} - \int \frac{\sin (n-2)x dx}{\cos^m x}.$$

$$373. \int \frac{(\cos px + i \sin px) dx}{\cos nx} = -2i \int \frac{z^{p+n-1} dz}{1+z^{2n}},$$

where $z = \cos x + i \sin x$. This yields two real integrals.

$$374. \int \frac{(\cos px + i \sin px) dx}{\sin nx} = -2 \int \frac{z^{p+n-1} dz}{1-z^{2n}},$$

where $z = \cos x + i \sin x$. This yields two real integrals.

$$375. \int \frac{(i \cos x - \sin x) dx}{\sqrt[n]{\cos nx}} = \int \frac{dy}{2 - y^n},$$

where $y = \frac{\cos x + i \sin x}{\sqrt[n]{\cos nx}}$. This yields two real integrals.

$$376. \int \sin ax \sin bx \sin cx dx = -\frac{1}{4} \left\{ \frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c} \right\}.$$

$$377. \int \cos ax \cos bx \cos cx dx = \frac{1}{4} \left\{ \frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a-b+c)x}{a-b+c} + \frac{\sin(a+b-c)x}{a+b-c} \right\}.$$

$$378. \int \sin ax \cos bx \cos cx dx = -\frac{1}{4} \left\{ \frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b} \right\}.$$

$$379. \int \cos ax \sin bx \sin cx dx = \frac{1}{4} \left\{ \frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a-b+c)x}{a-b+c} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a} \right\}$$

$$380. \int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2}.$$

$$381. \int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1-x^2}.$$

$$382. \int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \log(1+x^2).$$

$$383. \int \operatorname{ctn}^{-1} x dx = x \operatorname{ctn}^{-1} x + \frac{1}{2} \log(1+x^2).$$

$$384. \int \sec^{-1} x dx = x \sec^{-1} x - \log(x + \sqrt{x^2 - 1}).$$

$$385. \int \csc^{-1} x dx = x \csc^{-1} x + \log(x + \sqrt{x^2 - 1}).$$

$$386. \int \operatorname{versin}^{-1} x dx = (x - 1) \operatorname{versin}^{-1} x + \sqrt{2x - x^2}.$$

$$387. \int (\sin^{-1} x)^2 dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2} \sin^{-1} x.$$

$$388. \int (\cos^{-1} x)^2 dx = x(\cos^{-1} x)^2 - 2x - 2\sqrt{1 - x^2} \cos^{-1} x.$$

$$389. \int x \sin^{-1} x dx = \frac{1}{4}[(2x^2 - 1) \sin^{-1} x + x\sqrt{1 - x^2}].$$

$$390. \int x \cos^{-1} x dx = \frac{1}{4}[(2x^2 - 1) \cos^{-1} x - x\sqrt{1 - x^2}].$$

$$391. \int x \tan^{-1} x dx = \frac{1}{2}[(x^2 + 1) \tan^{-1} x - x].$$

$$392. \int x \operatorname{ctn}^{-1} x dx = \frac{1}{2}[(x^2 + 1) \operatorname{ctn}^{-1} x + x].$$

$$393. \int x \sec^{-1} x dx = \frac{1}{2}[x^2 \sec^{-1} x - \sqrt{x^2 - 1}].$$

$$394. \int x \csc^{-1} x dx = \frac{1}{2}[x^2 \csc^{-1} x + \sqrt{x^2 - 1}].$$

$$395. \int x^n \sin^{-1} x dx = \frac{1}{n+1} \left(x^{n+1} \sin^{-1} x - \int \frac{x^{n+1} dx}{\sqrt{1 - x^2}} \right).$$

$$396. \int x^n \cos^{-1} x dx = \frac{1}{n+1} \left(x^{n+1} \cos^{-1} x + \int \frac{x^{n+1} dx}{\sqrt{1 - x^2}} \right).$$

$$\int_{-1}^1 e^{xy} dy = \frac{e^x - e^{-x}}{x}$$

$$397. \int x^n \tan^{-1} x dx = \frac{1}{n+1} \left(x^{n+1} \tan^{-1} x - \int \frac{x^{n+1} dx}{1+x^2} \right).$$

$$398. \int x^n \cot^{-1} x dx = \frac{1}{n+1} \left(x^{n+1} \cot^{-1} x + \int \frac{x^{n+1} dx}{1+x^2} \right).$$

$$399. \int \frac{\sin^{-1} x dx}{x^2} = \log \left(\frac{1 - \sqrt{1-x^2}}{x} \right) - \frac{\sin^{-1} x}{x}.$$

$$400. \int \frac{\tan^{-1} x dx}{x^2} = \log x - \frac{1}{2} \log(1+x^2) - \frac{\tan^{-1} x}{x}.$$

$$401. \int e^{ax} dx = \frac{e^{ax}}{a}. \quad \int f(e^{ax}) dx = \int \frac{f(y) dy}{ay}, \quad y = e^{ax}.$$

$$402. \int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1).$$

$$403. \int x^m e^{ax} dx = \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} dx.$$

$$404. \int \frac{e^{ax}}{x^m} dx = \frac{1}{m-1} \left[-\frac{e^{ax}}{x^{m-1}} + a \int \frac{e^{ax} dx}{x^{m-1}} \right].$$

$$405. \int a^{bx} dx = \frac{a^{bx}}{b \log a}. \quad \int f(a^{bx}) dx = \int \frac{f(y) dy}{b \cdot \log a \cdot y}, \quad y = a^{bx}.$$

$$406. \int x^n a^x dx = \frac{a^x x^n}{\log a} - \frac{n a^x x^{n-1}}{(\log a)^2} + \frac{n(n-1) a^x x^{n-2}}{(\log a)^3} \dots \\ \pm \frac{n(n-1)(n-2) \dots 2.1 a^x}{(\log a)^{n+1}}.$$

$$407. \int \frac{a^x dx}{x^n} = \frac{1}{n-1} \left[-\frac{a^x}{x^{n-1}} - \frac{a^x \cdot \log a}{(n-2) x^{n-2}} \right. \\ \left. - \frac{a^x \cdot (\log a)^2}{(n-2)(n-3) x^{n-3}} - \dots + \frac{(\log a)^{n-1}}{(n-2)(n-3) \dots 2.1} \int \frac{a^x dx}{x} \right].$$

$$408. \int \frac{a^x dx}{x} = \log x + x \log a + \frac{(x \log a)^2}{2 \cdot 2!} + \frac{(x \log a)^3}{3 \cdot 3!} + \dots.$$

$$409. \int \frac{dx}{1+e^x} = \log \frac{e^x}{1+e^x}.$$

$$410. \int \frac{dx}{a+be^{mx}} = \frac{1}{am} [mx - \log(a+be^{mx})].$$

$$411. \int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}} \tan^{-1} \left(e^{mx} \sqrt{\frac{a}{b}} \right).$$

$$412. \int \frac{dx}{\sqrt{a+be^{mx}}} = \frac{1}{m\sqrt{a}} \{ \log(\sqrt{a+be^{mx}} - \sqrt{a}) \\ - \log(\sqrt{a+be^{mx}} + \sqrt{a}) \}, \text{ or } \frac{2}{m\sqrt{-a}} \tan^{-1} \frac{\sqrt{a+be^{mx}}}{\sqrt{-a}}.$$

$$413. \int \frac{xe^x dx}{(1+x)^2} = \frac{e^x}{1+x}, \quad \int x^n \cdot e^{ax^{n+1}} dx = \frac{e^{ax^{n+1}}}{a(n+1)}.$$

$$414. \int e^{ax} \sin px \, dx = \frac{e^{ax}(a \sin px - p \cos px)}{a^2 + p^2}.$$

$$415. \int e^{ax} \cos px \, dx = \frac{e^{ax}(a \cos px + p \sin px)}{a^2 + p^2}.$$

$$416. \int e^{ax} \log x \, dx = \frac{e^{ax} \log x}{a} - \frac{1}{a} \int \frac{e^{ax} dx}{x}.$$

$$417. \int e^{ax} \sin^2 x \, dx = \frac{e^{ax}}{4+a^2} \left(\sin x (a \sin x - 2 \cos x) + \frac{2}{a} \right).$$

$$418. \int e^{ax} \cos^2 x \, dx = \frac{e^{ax}}{4+a^2} \left(\cos x (2 \sin x + a \cos x) + \frac{2}{a} \right).$$

$$419. \int e^{ax} \sin^n bx \, dx = \frac{1}{a^2 + n^2 b^2} \left((a \sin bx \right. \\ \left. - nb \cos bx) e^{ax} \sin^{n-1} bx + n(n-1)b^2 \int e^{ax} \sin^{n-2} bx \cdot dx \right).$$

$$420. \int e^{ax} \cos^n bx dx = \frac{1}{a^2 + n^2 b^2} \left((a \cos bx + nb \sin bx) e^{ax} \cos^{n-1} bx + n(n-1)b^2 \int e^{ax} \cos^{n-2} bx dx \right).$$

$$421. \int e^{ax} \tan^n x dx = \frac{e^{ax} \tan^{n-1} x}{n-1} - \frac{a}{n-1} \int e^{ax} \tan^{n-1} x dx - \int e^{ax} \tan^{n-2} x dx.$$

$$422. \int e^{ax} \operatorname{ctn}^n x dx = -\frac{e^{ax} \operatorname{ctn}^{n-1} x}{n-1} + \frac{a}{n-1} \int e^{ax} \operatorname{ctn}^{n-1} x dx - \int e^{ax} \operatorname{ctn}^{n-2} x dx.$$

$$423. \int \frac{e^{ax} dx}{\sin^n x} = -e^{ax} \frac{a \sin x + (n-2) \cos x}{(n-1)(n-2) \sin^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax} dx}{\sin^{n-2} x}.$$

$$424. \int \frac{e^{ax} dx}{\cos^n x} = -e^{ax} \frac{a \cos x - (n-2) \sin x}{(n-1)(n-2) \cos^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax} dx}{\cos^{n-2} x}.$$

$$425. \int e^{ax} \sin^m x \cos^n x dx = \frac{1}{(m+n)^2 + a^2} \left\{ e^{ax} \sin^m x \cos^{n-1} x (a \cos x + (m+n) \sin x) - ma \int e^{ax} \sin^{m-1} x \cos^{n-1} x dx + (n-1)(m+n) \int e^{ax} \sin^m x \cos^{n-2} x dx \right\}$$

$$\begin{aligned}
&= \frac{1}{(m+n)^2 + a^2} \left\{ e^{ax} \sin^{m-1} x \cos^n x (a \sin x - (m+n) \cos x) \right. \\
&\quad + na \int e^{ax} \sin^{m-1} x \cos^{n-1} x dx \\
&\quad \left. + (m-1)(m+n) \int e^{ax} \sin^{m-2} x \cos^n x dx \right\} \\
&= \frac{1}{(m+n)^2 + a^2} \left\{ [e^{ax} \cos^{n-1} x \sin^{m-1} x (a \sin x \cos x + n \sin^2 x \right. \\
&\quad \left. - m \cos^2 x)] + n(n-1) \int e^{ax} \sin^m x \cos^{n-2} x dx \right. \\
&\quad \left. + m(m-1) \int e^{ax} \sin^{m-2} x \cos^n x dx \right\} \\
&= \frac{1}{(m+n)^2 + a^2} \left\{ [e^{ax} \sin^{m-1} x \cos^{n-1} x (a \sin x \cos x + n \sin^2 x \right. \\
&\quad \left. - m \cos^2 x)] + n(n-1) \int e^{ax} \sin^{m-2} x \cos^{n-2} x dx \right. \\
&\quad \left. + (m-n)(m+n-1) \int e^{ax} \sin^{m-2} x \cos^n x dx \right\} \\
&= \frac{1}{(m+n)^2 + a^2} \left\{ [e^{ax} \sin^{m-1} x \cos^{n-1} x (a \sin x \cos x + n \sin^2 x \right. \\
&\quad \left. - m \cos^2 x)] + m(m-1) \int e^{ax} \sin^{m-2} x \cos^{n-2} x dx \right. \\
&\quad \left. - (m-n)(m+n-1) \int e^{ax} \sin^m x \cos^{n-1} x dx \right\}.
\end{aligned}$$

$$426. \int \log x dx = x \log x - x.$$

$$427. \int x^m \log x dx = x^{m+1} \left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2} \right].$$

$$428. \int (\log x)^n dx = x (\log x)^n - n \int (\log x)^{n-1} dx.$$

$$429. \int x^m (\log x)^n dx = \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx.$$

$$430. \int \frac{(\log x)^n dx}{x} = \frac{(\log x)^{n+1}}{n+1}.$$

$$431. \int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2 \cdot 2!} + \frac{(\log x)^3}{3 \cdot 3!} + \dots$$

$$432. \int \frac{dx}{(\log x)^n} = -\frac{x}{(n-1)(\log x)^{n-1}} + \frac{1}{n-1} \int \frac{dx}{(\log x)^{n-1}}.$$

$$433. \int \frac{x^m dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m dx}{(\log x)^{n-1}}.$$

$$434. \int \frac{x^m dx}{\log x} = \int \frac{e^{-y}}{y} dy, \text{ where } y = -(m+1) \log x.$$

$$435. \int \frac{dx}{x \log x} = \log(\log x), \text{ and } \int \frac{(n-1) dx}{x(\log x)^n} = \frac{-1}{(\log x)^{n-1}}.$$

$$436. \int \log(a^2 + x^2) dx = x \cdot \log(a^2 + x^2) - 2x + 2a \cdot \tan^{-1}\left(\frac{x}{a}\right)$$

$$437. \int (a + bx)^m \log x dx \\ = \frac{1}{b(m+1)} \left[(a + bx)^{m+1} \log x - \int \frac{(a + bx)^{m+1} dx}{x} \right].$$

$$438. \int x^m \log(a + bx) dx \\ = \frac{1}{m+1} \left[x^{m+1} \log(a + bx) - b \int \frac{x^{m+1} dx}{a + bx} \right].$$

$$439. \int \frac{\log(a + bx) dx}{x} \\ = \log a \cdot \log x + \frac{bx}{a} - \frac{1}{2^2} \left(\frac{bx}{a}\right)^2 + \frac{1}{3^2} \left(\frac{bx}{a}\right)^3 - \dots \\ = \frac{1}{2} (\log bx)^2 - \frac{a}{bx} + \frac{1}{2^2} \left(\frac{a}{bx}\right)^2 - \frac{1}{3^2} \left(\frac{a}{bx}\right)^3 + \dots$$

$$440. \int \frac{\log x \, dx}{(a + bx)^m}$$

$$= \frac{1}{b(m-1)} \left[-\frac{\log x}{(a + bx)^{m-1}} + \int \frac{dx}{x(a + bx)^{m-1}} \right].$$

$$441. \int \frac{\log x \, dx}{a + bx} = \frac{1}{b} \log x \cdot \log(a + bx) - \frac{1}{b} \int \frac{\log(a + bx) \, dx}{x}.$$

$$442. \int (a + bx) \log x \, dx = \frac{(a + bx)^2}{2b} \log x - \frac{a^2 \log x}{2b} - ax - \frac{1}{4}bx^2.$$

$$443. \int \frac{\log x \, dx}{\sqrt{a + bx}}$$

$$= \frac{2}{b} \left[(\log x - 2) \sqrt{a + bx} + \sqrt{a} \log(\sqrt{a + bx} + \sqrt{a}) \right. \\ \left. - \sqrt{a} \log(\sqrt{a + bx} - \sqrt{a}) \right], \text{ if } a > 0$$

$$= \frac{2}{b} \left[(\log x - 2) \sqrt{a + bx} + 2\sqrt{-a} \tan^{-1} \sqrt{\frac{a + bx}{-a}} \right], \text{ if } a < 0.$$

$$444. \int \sin \log x \, dx = \frac{1}{2} x [\sin \log x - \cos \log x].$$

$$445. \int \cos \log x \, dx = \frac{1}{2} x [\sin \log x + \cos \log x].$$

$$446. \int \sinh x \, dx = \cosh x.$$

$$447. \int \cosh x \, dx = \sinh x.$$

$$448. \int \tanh x \, dx = \log \cosh x.$$

$$449. \int \operatorname{ctnh} x \, dx = \log \sinh x.$$

$$450. \int \operatorname{sech} x dx = 2 \tan^{-1} e^x.$$

$$451. \int \operatorname{csch} x dx = \log \tanh \frac{x}{2}.$$

$$\begin{aligned} 452. \int \sinh^n x dx &= \frac{1}{n} \sinh^{n-1} x \cdot \cosh x - \frac{n-1}{n} \int \sinh^{n-2} x dx \\ &= \frac{1}{n+1} \sinh^{n+1} x \cosh x - \frac{n+2}{n+1} \int \sinh^{n+2} x dx. \end{aligned}$$

$$\begin{aligned} 453. \int \cosh^n x dx &= \frac{1}{n} \sinh x \cdot \cosh^{n-1} x + \frac{n-1}{n} \int \cosh^{n-2} x dx \\ &= -\frac{1}{n+1} \sinh x \cosh^{n+1} x + \frac{n+2}{n+1} \int \cosh^{n+2} x dx. \end{aligned}$$

$$454. \int x \sinh x dx = x \cosh x - \sinh x.$$

$$455. \int x \cosh x dx = x \sinh x - \cosh x.$$

$$456. \int x^2 \sinh x dx = (x^2 + 2) \cosh x - 2x \sinh x.$$

$$\begin{aligned} 457. \int x^n \sinh x dx &= x^n \cosh x - nx^{n-1} \sinh x \\ &\quad + n(n-1) \int x^{n-2} \sinh x dx. \end{aligned}$$

$$458. \int \sinh^2 x dx = \frac{1}{2} (\sinh x \cosh x - x).$$

$$459. \int \sinh x \cdot \cosh x dx = \frac{1}{4} \cosh (2x).$$

$$460. \int \cosh^2 x dx = \frac{1}{2} (\sinh x \cosh x + x).$$

$$461. \int \tanh^2 x dx = x - \tanh x.$$

$$462. \int \operatorname{ctnh}^2 x \, dx = x - \operatorname{ctnh} x.$$

$$463. \int \operatorname{sech}^2 x \, dx = \tanh x.$$

$$464. \int \operatorname{csch}^2 x \, dx = -\operatorname{ctnh} x.$$

$$465. \int \sinh^{-1} x \, dx = x \sinh^{-1} x - \sqrt{1+x^2}.$$

$$466. \int \cosh^{-1} x \, dx = x \cosh^{-1} x - \sqrt{x^2-1}.$$

$$467. \int \tanh^{-1} x \, dx = x \tanh^{-1} x + \frac{1}{2} \log(1-x^2).$$

$$468. \int x \sinh^{-1} x \, dx = \frac{1}{4} [(2x^2+1) \sinh^{-1} x - x \sqrt{1+x^2}].$$

$$469. \int x \cosh^{-1} x \, dx = \frac{1}{4} [(2x^2-1) \cosh^{-1} x - x \sqrt{x^2-1}].$$

$$\begin{aligned} 470. \int \frac{dx}{\cosh a + \cosh x} \\ &= \operatorname{csch} a [\log \cosh \tfrac{1}{2}(x+a) - \log \cosh \tfrac{1}{2}(x-a)], \\ &= 2 \operatorname{csch} a \cdot \tanh^{-1} (\tanh \tfrac{1}{2} x \cdot \tanh \tfrac{1}{2} a). \end{aligned}$$

$$471. \int \frac{dx}{\cos a + \cosh x} = 2 \operatorname{csc} a \cdot \tan^{-1} (\tanh \tfrac{1}{2} x \cdot \tan \tfrac{1}{2} a).$$

$$472. \int \frac{dx}{1 + \cos a \cdot \cosh x} = 2 \operatorname{csc} a \cdot \tanh^{-1} (\tanh \tfrac{1}{2} x \cdot \tan \tfrac{1}{2} a).$$

$$473. \int \sinh x \cdot \cos x \, dx = \tfrac{1}{2} (\cosh x \cdot \cos x + \sinh x \cdot \sin x).$$

$$474. \int \cosh x \cdot \cos x \, dx = \tfrac{1}{2} (\sinh x \cdot \cos x + \cosh x \cdot \sin x).$$

$$475. \int \sinh x \cdot \sin x \, dx = \tfrac{1}{2} (\cosh x \cdot \sin x - \sinh x \cdot \cos x).$$

$$476. \int \cosh x \cdot \sin x \, dx = \frac{1}{2} (\sinh x \cdot \sin x - \cosh x \cdot \cos x).$$

$$477. \int \sinh (mx) \sinh (nx) \, dx \\ = \frac{1}{m^2 - n^2} \left[m \sinh (nx) \cosh (mx) - n \cosh (nx) \sinh (mx) \right].$$

$$478. \int \cosh (mx) \sinh (nx) \, dx \\ = \frac{1}{m^2 - n^2} \left[m \sinh (nx) \sinh (mx) - n \cosh (nx) \cosh (mx) \right].$$

$$479. \int \cosh (mx) \cosh (nx) \, dx \\ = \frac{1}{m^2 - n^2} \left[m \sinh (mx) \cosh (nx) - n \sinh (nx) \cosh (mx) \right].$$

$$\int \frac{dx}{a \cos^2 x + c \sin x \cdot \cos x + b \sin^2 x} = \int \frac{d(\tan x)}{a + c \tan x + b \tan^2 x} \\ \int \frac{(l + m \cos x + n \sin x) \, dx}{a + b \cos x + c \sin x} = \int \frac{(m \cos \delta + n \sin \delta) \cos z \cdot dz}{Z} \\ + \int \frac{l \cdot dz}{Z} - \int \frac{(m \sin \delta - n \cos \delta) \sin z \cdot dz}{Z},$$

where $b = q \cdot \cos \delta$, $c = q \cdot \sin \delta$, $z = x - \delta$, $Z = a + q \cdot \cos z$.

$$\int \sin (mx + a) \cdot \sin (nx + b) \, dx \quad [\text{See 303 and 304.}]$$

$$= \frac{\sin [mx - nx + a - b]}{2(m - n)} - \frac{\sin [mx + nx + a + b]}{2(m + n)}.$$

$$\int \cos (mx + a) \cdot \cos (nx + b) \, dx$$

$$= \frac{\sin [mx + nx + a + b]}{2(m + n)} + \frac{\sin [mx - nx + a - b]}{2(m - n)}.$$

$$\int \sin (mx + a) \cdot \cos (nx + b) \, dx$$

$$= -\frac{\cos [mx + nx + a + b]}{2(m + n)} - \frac{\cos [mx - nx + a - b]}{2(m - n)}.$$

VI. MISCELLANEOUS DEFINITE INTEGRALS.*

$$480. \int_0^{\infty} \frac{a dx}{a^2 + x^2} = \frac{\pi}{2}, \text{ if } a > 0; 0, \text{ if } a = 0; -\frac{\pi}{2}, \text{ if } a < 0.$$

$$481. \int_0^{\infty} x^{n-1} e^{-x} dx = \int_0^1 \left[\log \frac{1}{x} \right]^{n-1} dx \equiv \Gamma(n).$$

$$\Gamma(z+1) = z \cdot \Gamma(z), \text{ if } z > 0.$$

$$\Gamma(y) \cdot \Gamma(1-y) = \frac{\pi}{\sin \pi y}, \text{ if } 1 > y > 0. \quad \Gamma(2) = \Gamma(1) = 1.$$

$$\Gamma(n+1) = n!, \text{ if } n \text{ is an integer.} \quad \Gamma(z) = \Pi(z-1).$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \quad Z(y) = D_y[\log \Gamma(y)]. \quad Z(1) = -0.577216.$$

$$482. \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^{\infty} \frac{x^{m-1} dx}{(1+x)^{m+n}} = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.$$

$$483. \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots (n)} \cdot \frac{\pi}{2}, \text{ if } n \text{ is an even integer,}$$

$$= \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdot 7 \cdots n}, \text{ if } n \text{ is an odd integer,}$$

$$= \frac{1}{2} \sqrt{\pi} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}, \text{ for any value of } n \text{ greater than } -1.$$

$$484. \int_0^{\infty} \frac{\sin mx dx}{x} = \frac{\pi}{2}, \text{ if } m > 0; 0, \text{ if } m = 0; -\frac{\pi}{2}, \text{ if } m < 0.$$

* For very complete lists of definite integrals, see Bierens de Haan, *Tables d'intégrales définies*, Amsterdam, 1858-64, and *Nouv. Tables d'intégrales définies*, Leyden, 1867.

$$485. \int_0^{\infty} \frac{\sin x \cdot \cos mx \, dx}{x} = 0, \text{ if } m < -1 \text{ or } m > 1;$$

$$\frac{\pi}{4}, \text{ if } m = -1 \text{ or } m = 1; \frac{\pi}{2}, \text{ if } -1 < m < 1.$$

$$486. \int_0^{\infty} \frac{\sin^2 x \, dx}{x^2} = \frac{\pi}{2}.$$

$$487. \int_0^{\infty} \cos(x^2) \, dx = \int_0^{\infty} \sin(x^2) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}.$$

$$488. \int_0^{\pi} \sin kx \cdot \sin mx \, dx = \int_0^{\pi} \cos kx \cdot \cos mx \, dx = 0,$$

if k is different from m .

$$489. \int_0^{\pi} \sin^2 mx \, dx = \int_0^{\pi} \cos^2 mx \, dx = \frac{\pi}{2}.$$

$$490. \int_0^{\infty} \frac{\cos mx \, dx}{1+x^2} = \frac{\pi}{2} \cdot e^{-m}. \quad m > 0.$$

$$491. \int_0^{\infty} \frac{\cos x \, dx}{\sqrt{x}} = \int_0^{\infty} \frac{\sin x \, dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}.$$

$$492. \int_0^{\infty} e^{-a^2 x^2} \, dx = \frac{1}{2a} \sqrt{\pi} = \frac{1}{2a} \Gamma\left(\frac{1}{2}\right).$$

$$493. \int_0^{\infty} x^n e^{-ax} \, dx = \frac{\Gamma(n+1)}{a^{n+1}} = \frac{n!}{a^{n+1}}.$$

$$494. \int_0^{\infty} x^{2n} e^{-ax^2} \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}.$$

$$495. \int_0^{\infty} e^{-x^2 - \frac{a^2}{x^2}} \, dx = \frac{e^{-2a} \sqrt{\pi}}{2}. \quad a > 0.$$

$$496. \int_0^{\infty} e^{-nx} \sqrt{x} \, dx = \frac{1}{2n} \sqrt{\frac{\pi}{n}}.$$

$$497. \int_0^{\infty} \frac{e^{-nx}}{\sqrt{x}} \, dx = \sqrt{\frac{\pi}{n}}. \quad a > 0.$$

$$498. \int_0^{\infty} \frac{dx}{e^{nx} + e^{-nx}} = \frac{\pi}{4n}.$$

$$499. \int_0^{\infty} \frac{x dx}{e^{nx} - e^{-nx}} = \frac{\pi^2}{8n^2}.$$

$$500. \int_0^{\pi i} \sinh(mx) \cdot \sinh(nx) dx = \int_0^{\pi i} \cosh(mx) \cdot \cosh(nx) dx \\ = 0, \text{ if } m \text{ is different from } n.$$

$$501. \int_0^{\pi i} \cosh^2(mx) dx = - \int_0^{\pi i} \sinh^2(mx) dx = \frac{\pi i}{2}.$$

$$502. \int_{-\pi i}^{+\pi i} \sinh(mx) dx = 0.$$

$$503. \int_0^{\pi i} \cosh(mx) dx = 0.$$

$$504. \int_{-\pi i}^{\pi i} \sinh(mx) \cosh(nx) dx = 0.$$

$$505. \int_0^{\pi i} \sinh(mx) \cosh(mx) dx = 0.$$

$$506. \int_0^{\infty} e^{-ax} \cos mx dx = \frac{a}{a^2 + m^2}, \text{ if } a > 0.$$

$$507. \int_0^{\infty} e^{-ax} \sin mx dx = \frac{m}{a^2 + m^2}, \text{ if } a > 0.$$

$$508. \int_0^{\infty} e^{-a^2 x^2} \cos bx dx = \frac{\sqrt{\pi} \cdot e^{-\frac{b^2}{4a^2}}}{2a}. \quad a > 0.$$

$$509. \int_0^1 \frac{\log x}{1-x} dx = -\frac{\pi^2}{6}.$$

$$510. \int_0^1 \frac{\log x}{1+x} dx = -\frac{\pi^2}{12}.$$

$$511. \int_0^1 \frac{\log x}{1-x^2} dx = -\frac{\pi^2}{8}.$$

$$512. \int_0^1 \log \left(\frac{1+x}{1-x} \right) \cdot \frac{dx}{x} = \frac{\pi^2}{4}.$$

$$513. \int_0^1 \frac{\log x dx}{\sqrt{1-x^2}} = -\frac{\pi}{2} \log 2.$$

$$514. \int_0^1 \frac{(x^p - x^q) dx}{\log x} = \log \frac{p+1}{q+1}, \text{ if } p+1 > 0, q+1 > 0.$$

$$515. \int_0^1 (\log x)^n dx = (-1)^n \cdot n!.$$

$$516. \int_0^1 \left(\log \frac{1}{x} \right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}.$$

$$517. \int_0^1 \left(\log \frac{1}{x} \right)^n dx = n!.$$

$$518. \int_0^1 \frac{dx}{\sqrt{\log \left(\frac{1}{x} \right)}} = \sqrt{\pi}.$$

$$519. \int_0^1 x^m \log \left(\frac{1}{x} \right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \text{ if } m+1 > 0, n+1 > 0.$$

$$520. \int_0^\infty \log \left(\frac{e^x + 1}{e^x - 1} \right) dx = \frac{\pi^2}{4}.$$

$$521. \int_0^{\frac{\pi}{2}} \log \sin x dx = \int_0^{\frac{\pi}{2}} \log \cos x dx = -\frac{\pi}{2} \cdot \log 2.$$

$$522. \int_0^\pi x \cdot \log \sin x dx = -\frac{\pi^2}{2} \log 2.$$

$$523. \int_0^\pi \log (a \pm b \cos x) dx = \pi \log \left(\frac{a + \sqrt{a^2 - b^2}}{2} \right). \quad a \geq b.$$

VII. ELLIPTIC INTEGRALS.

$$F(\phi, k) \equiv \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \equiv \int_0^x \frac{dz}{\sqrt{1 - z^2} \sqrt{1 - k^2 z^2}} \equiv u,$$

where $k^2 < 1$, $x = \sin \phi$.

$$E(\phi, k) \equiv \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} \cdot d\theta.$$

$$\Pi(\phi, n, k) \equiv \int_0^\phi \frac{d\theta}{(1 + n \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}}.$$

$$\phi \equiv \text{am } u, \sin \phi \equiv x \equiv \text{sn } u, \cos \phi \equiv \sqrt{1 - x^2} \equiv \text{cn } u, \tan \phi \equiv \text{tn } u,$$

$$\Delta \phi \equiv \sqrt{1 - k^2 \sin^2 \phi} \equiv \sqrt{1 - k^2 x^2} \equiv \text{dn } u, k'^2 \equiv 1 - k^2.$$

$$u \equiv \text{am}^{-1}(\phi, k) \equiv \text{sn}^{-1}(x, k) \equiv \text{cn}^{-1}(\sqrt{1 - x^2}, k) \\ \equiv \text{dn}^{-1}(\sqrt{1 - k^2 x^2}, k).$$

$$K \equiv F(\tfrac{1}{2}\pi, k), K' \equiv F(\tfrac{1}{2}\pi, k'), E \equiv E(\tfrac{1}{2}\pi, k), E' \equiv E(\tfrac{1}{2}\pi, k').$$

$$\text{If } k_0 = \frac{2k^{\frac{1}{2}}}{1+k} \text{ and } \tan \phi \equiv \frac{\sin 2\omega}{k + \cos 2\omega},$$

$$F(\phi, k) \equiv \frac{2}{1+k} F(\omega, k_0).$$

$$\begin{aligned} 524. & \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \\ &= \frac{\pi}{2} \left[1 + \left(\tfrac{1}{2}\right)^2 k^2 + \left(\tfrac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \dots \right], \text{ if } k^2 < 1, \\ &= K. \end{aligned}$$

$$\begin{aligned} 525. & \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} \cdot d\theta \\ &= \frac{\pi}{2} \left[1 - \left(\tfrac{1}{2}\right)^2 k^2 - \left(\tfrac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \left(\tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{k^6}{5} - \dots \right], \text{ if } k^2 < 1, \\ &= E. \end{aligned}$$

$$\begin{aligned}
 526. \int_0^\phi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} &= \frac{2}{\pi} \phi \cdot K - \sin \phi \cos \phi \left[\frac{1 \cdot 1}{2 \cdot 2} k^2 \right. \\
 &\quad \left. + \frac{1 \cdot 3}{2 \cdot 4} A_4 k^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} A_6 k^6 + \dots \right] \\
 &= F(\phi, k),
 \end{aligned}$$

where $A_4 \equiv \frac{1}{4} \sin^2 \phi + \frac{3}{2 \cdot 4}$, $A_6 \equiv \frac{1}{6} \sin^4 \phi + \frac{5}{6 \cdot 4} \sin^2 \phi + \frac{5 \cdot 3}{6 \cdot 4 \cdot 2}$,
 $A_8 \equiv \frac{1}{8} \sin^6 \phi + \frac{7}{8 \cdot 6} \sin^4 \phi + \frac{7 \cdot 5}{8 \cdot 6 \cdot 4} \sin^2 \phi + \frac{7 \cdot 5 \cdot 3}{8 \cdot 6 \cdot 4 \cdot 2}$, etc.

$$\begin{aligned}
 527. \int_0^\phi \sqrt{1-k^2 \sin^2 \theta} \cdot d\theta &= \frac{2}{\pi} \phi \cdot E + \sin \phi \cos \phi \left[\frac{1 \cdot 1}{2 \cdot 2} k^2 \right. \\
 &\quad \left. + \frac{1}{2 \cdot 4} k^4 A_4 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} k^6 A_6 + \dots \right] \\
 &= E(\phi, k).
 \end{aligned}$$

$$\begin{aligned}
 528.* \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2 x^2)}} &= \text{sn}^{-1}(x, k) \\
 &= F(\sin^{-1} x, k). \quad 0 < x < 1.
 \end{aligned}$$

$$\begin{aligned}
 529. \int_x^1 \frac{dx}{\sqrt{(1-x^2)(k'^2 + k^2 x^2)}} &= \text{cn}^{-1}(x, k) \\
 &= F(\cos^{-1} x, k) = \text{sn}^{-1}(\sqrt{1-x^2}, k). \quad 0 < x < 1.
 \end{aligned}$$

$$\begin{aligned}
 530. \int_x^1 \frac{dx}{\sqrt{(1-x^2)(x^2-k'^2)}} &= \text{dn}^{-1}(x, k) \\
 &= F(\Delta^{-1} x, k) = \text{sn}^{-1}\left(\frac{1}{k} \sqrt{1-x^2}, k\right). \quad 0 < x < 1.
 \end{aligned}$$

$$\begin{aligned}
 531. \int_0^x \frac{dx}{\sqrt{(1+x^2)(1+k'^2 x^2)}} &= \text{tn}^{-1}(x, k) \\
 &= F(\tan^{-1} x, k) = \text{sn}^{-1}\left(\frac{x}{\sqrt{1+x^2}}, k\right). \quad 0 < x < 1.
 \end{aligned}$$

* The next forty-two integrals are copied in order from a class-room list of Prof. W. E. Byerly.

$$\begin{aligned}
 532. \int_0^x \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} &= 2 \operatorname{sn}^{-1}(\sqrt{x}, k) \\
 &= 2 F(\sin^{-1} \sqrt{x}, k). \quad 0 < x < 1.
 \end{aligned}$$

$$\begin{aligned}
 533. \int_x^1 \frac{dx}{\sqrt{x(1-x)(k'^2+k^2x)}} &= 2 \operatorname{cn}^{-1}(\sqrt{x}, k) \\
 &= 2 F(\cos^{-1} \sqrt{x}, k) = 2 \operatorname{sn}^{-1}(\sqrt{1-x}, k). \quad 0 < x < 1.
 \end{aligned}$$

$$\begin{aligned}
 534. \int_x^1 \frac{dx}{\sqrt{x(1-x)(x-k'^2)}} &= 2 \operatorname{dn}^{-1}(\sqrt{x}, k) \\
 &= 2 F(\Delta^{-1} \sqrt{x}, k) = 2 \operatorname{sn}^{-1}\left(\frac{1}{k} \sqrt{1-x}, k\right). \quad 0 < x < 1.
 \end{aligned}$$

$$\begin{aligned}
 535. \int_0^x \frac{dx}{\sqrt{(1+x)(1+k'^2x)}} &= 2 \operatorname{tn}^{-1}(\sqrt{x}, k) \\
 &= 2 F(\tan^{-1} \sqrt{x}, k) = 2 \operatorname{sn}^{-1}\left(\sqrt{\frac{x}{1+x}}, k\right). \quad 0 < x < 1.
 \end{aligned}$$

$$536. \int_0^x \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a} \operatorname{sn}^{-1}\left(\frac{x}{b}, \frac{b}{a}\right). \quad a > b > x > 0.$$

$$537. \int_x^\infty \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{a} \operatorname{sn}^{-1}\left(\frac{a}{x}, \frac{b}{a}\right). \quad x > a > b.$$

$$\begin{aligned}
 538. \int_x^b \frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} \\
 &= \frac{1}{\sqrt{a^2+b^2}} \operatorname{cn}^{-1}\left(\frac{x}{b}, \frac{b}{\sqrt{a^2+b^2}}\right). \quad b > x > 0.
 \end{aligned}$$

$$\begin{aligned}
 539. \int_b^x \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)}} \\
 &= \frac{1}{\sqrt{a^2+b^2}} \operatorname{cn}^{-1}\left(\frac{b}{x}, \frac{a}{\sqrt{a^2+b^2}}\right). \quad x > b > 0.
 \end{aligned}$$

$$\begin{aligned}
 540. \int_x^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} \\
 = \frac{1}{a} \operatorname{sn}^{-1} \left(\sqrt{\frac{a^2 - x^2}{a^2 - b^2}}, \sqrt{\frac{a^2 - b^2}{a^2}} \right). \quad a > x > b.
 \end{aligned}$$

$$\begin{aligned}
 541. \int_0^x \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} \\
 = \frac{1}{a} \operatorname{tn}^{-1} \left(\frac{x}{b}, \sqrt{\frac{a^2 - b^2}{a^2}} \right). \quad x > 0. \\
 a > \beta > \gamma.
 \end{aligned}$$

$$\begin{aligned}
 542. \int_x^\infty \frac{dx}{\sqrt{(x - a)(x - \beta)(x - \gamma)}} \\
 = \frac{2}{\sqrt{a - \gamma}} \operatorname{sn}^{-1} \left(\sqrt{\frac{a - \gamma}{x - \gamma}}, \sqrt{\frac{\beta - \gamma}{a - \gamma}} \right). \quad x > a.
 \end{aligned}$$

$$\begin{aligned}
 543. \int_a^x \frac{dx}{\sqrt{(x - a)(x - \beta)(x - \gamma)}} \\
 = \frac{2}{\sqrt{a - \gamma}} \operatorname{sn}^{-1} \left(\sqrt{\frac{x - a}{x - \beta}}, \sqrt{\frac{\beta - \gamma}{a - \gamma}} \right). \quad x > a.
 \end{aligned}$$

$$\begin{aligned}
 544. \int_x^a \frac{dx}{\sqrt{(a - x)(x - \beta)(x - \gamma)}} \\
 = \frac{2}{\sqrt{a - \gamma}} \operatorname{sn}^{-1} \left(\sqrt{\frac{a - x}{a - \beta}}, \sqrt{\frac{a - \beta}{a - \gamma}} \right). \quad a > x > \beta.
 \end{aligned}$$

$$\begin{aligned}
 545. \int_\beta^x \frac{dx}{\sqrt{(a - x)(x - \beta)(x - \gamma)}} \\
 = \frac{2}{\sqrt{a - \gamma}} \operatorname{sn}^{-1} \left(\sqrt{\frac{a - \gamma}{a - \beta} \cdot \frac{x - \beta}{x - \gamma}}, \sqrt{\frac{a - \beta}{a - \gamma}} \right). \quad a > x > \beta.
 \end{aligned}$$

$$\begin{aligned}
 546. \int_x^\beta \frac{dx}{\sqrt{(a - x)(\beta - x)(x - \gamma)}} \\
 = \frac{2}{\sqrt{a - \gamma}} \operatorname{sn}^{-1} \left(\sqrt{\frac{a - \gamma}{\beta - \gamma} \cdot \frac{\beta - x}{a - x}}, \sqrt{\frac{\beta - \gamma}{a - \gamma}} \right). \quad \beta > x > \gamma.
 \end{aligned}$$

$$\begin{aligned}
 547. \int_{\gamma}^x \frac{dx}{\sqrt{(a-x)(\beta-x)(x-\gamma)}} \\
 = \frac{2}{\sqrt{a-\gamma}} \operatorname{sn}^{-1} \left(\sqrt{\frac{x-\gamma}{\beta-\gamma}}, \sqrt{\frac{\beta-\gamma}{a-\gamma}} \right). \quad \beta > x > \gamma.
 \end{aligned}$$

$$\begin{aligned}
 548. \int_x^{\gamma} \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)}} \\
 = \frac{2}{\sqrt{a-\gamma}} \operatorname{sn}^{-1} \left(\sqrt{\frac{\gamma-x}{\beta-x}}, \sqrt{\frac{a-\beta}{a-\gamma}} \right); \quad \gamma > x.
 \end{aligned}$$

$$\begin{aligned}
 549. \int_{-x}^x \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)}} \\
 = \frac{2}{\sqrt{a-\gamma}} \operatorname{sn}^{-1} \left(\sqrt{\frac{a-\gamma}{a-x}}, \sqrt{\frac{a-\beta}{a-\gamma}} \right). \quad \gamma > x.
 \end{aligned}$$

$$a > \beta > \gamma > \delta.$$

$$\begin{aligned}
 550. \int_a^x \frac{dx}{\sqrt{(x-a)(x-\beta)(x-\gamma)(x-\delta)}} \\
 = \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1} \left(\sqrt{\frac{\beta-\delta}{a-\delta} \cdot \frac{x-a}{x-\beta}}, \sqrt{\frac{\beta-\gamma}{a-\gamma} \cdot \frac{a-\delta}{\beta-\delta}} \right). \\
 x > a.
 \end{aligned}$$

$$\begin{aligned}
 551. \int_x^a \frac{dx}{\sqrt{(a-x)(x-\beta)(x-\gamma)(x-\delta)}} \\
 = \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1} \left(\sqrt{\frac{\beta-\delta}{a-\beta} \cdot \frac{a-x}{x-\delta}}, \sqrt{\frac{a-\beta}{a-\gamma} \cdot \frac{\gamma-\delta}{\beta-\delta}} \right). \\
 a > x > \beta.
 \end{aligned}$$

$$\begin{aligned}
 552. \int_{\beta}^x \frac{dx}{\sqrt{(a-x)(x-\beta)(x-\gamma)(x-\delta)}} \\
 = \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1} \left(\sqrt{\frac{a-\gamma}{a-\beta} \cdot \frac{x-\beta}{x-\gamma}}, \sqrt{\frac{a-\beta}{a-\gamma} \cdot \frac{\gamma-\delta}{\beta-\delta}} \right). \\
 a > x > \beta.
 \end{aligned}$$

$$\begin{aligned}
 553. \quad & \int_x^\beta \frac{dx}{\sqrt{(a-x)(\beta-x)(x-\gamma)(x-\delta)}} \\
 &= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1} \left(\sqrt{\frac{a-\gamma}{\beta-\gamma} \cdot \frac{\beta-x}{a-x}}, \sqrt{\frac{\beta-\gamma}{a-\gamma} \cdot \frac{a-\delta}{\beta-\delta}} \right). \\
 & \qquad \qquad \qquad \beta > x > \gamma.
 \end{aligned}$$

$$\begin{aligned}
 554. \quad & \int_\gamma^x \frac{dx}{\sqrt{(a-x)(\beta-x)(x-\gamma)(x-\delta)}} \\
 &= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1} \left(\sqrt{\frac{\beta-\delta}{\beta-\gamma} \cdot \frac{x-\gamma}{x-\delta}}, \sqrt{\frac{\beta-\gamma}{a-\gamma} \cdot \frac{a-\delta}{\beta-\delta}} \right). \\
 & \qquad \qquad \qquad \beta > x > \gamma.
 \end{aligned}$$

$$\begin{aligned}
 555. \quad & \int_x^\gamma \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)(x-\delta)}} \\
 &= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1} \left(\sqrt{\frac{\beta-\delta}{\gamma-\delta} \cdot \frac{\gamma-x}{\beta-x}}, \sqrt{\frac{a-\beta}{a-\gamma} \cdot \frac{\gamma-\delta}{\beta-\delta}} \right). \\
 & \qquad \qquad \qquad \gamma > x > \delta.
 \end{aligned}$$

$$\begin{aligned}
 556. \quad & \int_\delta^x \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)(x-\delta)}} \\
 &= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1} \left(\sqrt{\frac{a-\gamma}{\gamma-\delta} \cdot \frac{x-\delta}{a-x}}, \sqrt{\frac{a-\beta}{a-\gamma} \cdot \frac{\gamma-\delta}{\beta-\delta}} \right). \\
 & \qquad \qquad \qquad \gamma > x > \delta.
 \end{aligned}$$

$$\begin{aligned}
 557. \quad & \int_x^\delta \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)(\delta-x)}} \\
 &= \frac{2}{\sqrt{(a-\gamma)(\beta-\delta)}} \operatorname{sn}^{-1} \left(\sqrt{\frac{a-\gamma}{a-\delta} \cdot \frac{\delta-x}{\gamma-x}}, \sqrt{\frac{\beta-\gamma}{a-\gamma} \cdot \frac{a-\delta}{\beta-\delta}} \right). \\
 & \qquad \qquad \qquad \delta > x.
 \end{aligned}$$

$$558. \quad \int \operatorname{sn} x \, dx = \frac{1}{k} \cosh^{-1} \left(\frac{\operatorname{dn} x}{k'} \right).$$

$$559. \quad \int \operatorname{cn} x \, dx = \frac{1}{k} \cos^{-1} (\operatorname{dn} x).$$

$$560. \int \operatorname{dn} x \, dx = \sin^{-1}(\operatorname{sn} x) = \operatorname{am} x.$$

$$561. \int \frac{dx}{\operatorname{sn} x} = \log \left[\frac{\operatorname{sn} x}{\operatorname{cn} x + \operatorname{dn} x} \right].$$

$$562. \int \frac{dx}{\operatorname{cn} x} = \frac{1}{k'} \log \left[\frac{k' \operatorname{sn} x + \operatorname{dn} x}{\operatorname{cn} x} \right].$$

$$563. \int \frac{dx}{\operatorname{dn} x} = \frac{1}{k'} \tan^{-1} \left[\frac{k' \operatorname{sn} x - \operatorname{cn} x}{k' \operatorname{sn} x + \operatorname{cn} x} \right].$$

$$564. \int_0^x \operatorname{sn}^2 x \, dx = \frac{1}{k^2} [x - E(\operatorname{am} x, k)].$$

$$565. \int_0^x \operatorname{cn}^2 x \, dx = \frac{1}{k^2} [E(\operatorname{am} x, k) - k'^2 x].$$

$$566. \int_0^x \operatorname{dn}^2 x \, dx = E(\operatorname{am} x, k).$$

$$567. (m+1) \int \operatorname{sn}^m x \, dx = (m+2)(1+k^2) \int \operatorname{sn}^{m+2} x \, dx \\ - (m+3)k^2 \int \operatorname{sn}^{m+4} x \, dx + \operatorname{sn}^{m+1} x \operatorname{cn} x \operatorname{dn} x.$$

$$568. (m+1)k'^2 \int \operatorname{cn}^m x \, dx = (m+2)(1-2k^2) \int \operatorname{cn}^{m+2} x \, dx \\ + (m+3)k^2 \int \operatorname{cn}^{m+4} x \, dx - \operatorname{cn}^{m+1} x \operatorname{sn} x \operatorname{dn} x.$$

$$569. (m+1)k'^2 \int \operatorname{dn}^m x \, dx = (m+2)(2-k^2) \int \operatorname{dn}^{m+2} x \, dx \\ - (m+3) \int \operatorname{dn}^{m+4} x \, dx + k^2 \operatorname{dn}^{m+1} x \operatorname{sn} x \operatorname{cn} x.$$

Since $\sin^2 \theta \equiv \frac{1}{k^2} - \frac{1}{k'^2} (1 - k^2 \sin^2 \theta)$,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta \cdot d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{1}{k^2} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} - \frac{1}{k'^2} \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} \cdot d\theta.$$

VIII. AUXILIARY FORMULAS.

A. — TRIGONOMETRIC FUNCTIONS.

$$570. \tan a \cdot \operatorname{ctn} a = \sin a \cdot \csc a = \cos a \cdot \sec a = 1,$$

$$\tan a = \sin a \div \cos a, \quad \sec^2 a = 1 + \tan^2 a,$$

$$\csc^2 a = 1 + \operatorname{ctn}^2 a, \quad \sin^2 a + \cos^2 a = 1.$$

$$571. \sin a = \sqrt{1 - \cos^2 a} = 2 \sin \frac{1}{2} a \cdot \cos \frac{1}{2} a = \cos a \cdot \tan a$$

$$= \frac{1}{\sqrt{1 + \operatorname{ctn}^2 a}} = \frac{\tan a}{\sqrt{1 + \tan^2 a}} = \sqrt{\frac{1 - \cos 2a}{2}} = \frac{2 \tan \frac{1}{2} a}{1 + \tan^2 \frac{1}{2} a}$$

$$= \sqrt{\frac{\sec^2 a - 1}{\sec^2 a}} = \operatorname{ctn} \frac{1}{2} a \cdot (1 - \cos a) = \tan \frac{1}{2} a \cdot (1 + \cos a).$$

$$572. \cos a = \sqrt{1 - \sin^2 a} = \frac{1}{\sqrt{1 + \tan^2 a}} = \frac{\operatorname{ctn} a}{\sqrt{1 + \operatorname{ctn}^2 a}}$$

$$= \sqrt{\frac{1 + \cos 2a}{2}} = \frac{1 - \tan^2 \frac{1}{2} a}{1 + \tan^2 \frac{1}{2} a} = \cos^2 \frac{1}{2} a - \sin^2 \frac{1}{2} a$$

$$= 1 - 2 \sin^2 \frac{1}{2} a = 2 \cos^2 \frac{1}{2} a - 1 = \sin a \cdot \operatorname{ctn} a$$

$$= \frac{\sin 2a}{2 \sin a} = \sqrt{\frac{\csc^2 a - 1}{\csc^2 a}} = \frac{\operatorname{ctn} \frac{1}{2} a - \tan \frac{1}{2} a}{\operatorname{ctn} \frac{1}{2} a + \tan \frac{1}{2} a}.$$

$$573. \tan a = \frac{\sin a}{\sqrt{1 - \sin^2 a}} = \frac{\sqrt{1 - \cos^2 a}}{\cos a} = \frac{\sin 2a}{1 + \cos 2a}$$

$$= \frac{1 - \cos 2a}{\sin 2a} = \sqrt{\frac{1 - \cos 2a}{1 + \cos 2a}} = \frac{2 \tan \frac{1}{2} a}{1 - \tan^2 \frac{1}{2} a}$$

$$= \frac{\sec a}{\csc a} = \frac{2}{\operatorname{ctn} \frac{1}{2} a - \tan \frac{1}{2} a} = \frac{2 \operatorname{ctn} \frac{1}{2} a}{\operatorname{ctn}^2 \frac{1}{2} a - 1}.$$

574.

| | $-\alpha$. | $90^\circ \pm \alpha$. | $180^\circ \pm \alpha$. | $270^\circ \pm \alpha$. | $360^\circ \pm \alpha$. |
|-----|----------------|-------------------------|--------------------------|--------------------------|--------------------------|
| sin | $-\sin \alpha$ | $+\cos \alpha$ | $\mp \sin \alpha$ | $-\cos \alpha$ | $\pm \sin \alpha$ |
| cos | $+\cos \alpha$ | $\mp \sin \alpha$ | $-\cos \alpha$ | $\pm \sin \alpha$ | $+\cos \alpha$ |
| tan | $-\tan \alpha$ | $\mp \cot \alpha$ | $\pm \tan \alpha$ | $\mp \cot \alpha$ | $\pm \tan \alpha$ |
| ctn | $-\cot \alpha$ | $\mp \tan \alpha$ | $\pm \cot \alpha$ | $\mp \tan \alpha$ | $\pm \cot \alpha$ |
| sec | $+\sec \alpha$ | $\mp \csc \alpha$ | $-\sec \alpha$ | $\pm \csc \alpha$ | $+\sec \alpha$ |
| csc | $-\csc \alpha$ | $+\sec \alpha$ | $\mp \csc \alpha$ | $-\sec \alpha$ | $\pm \csc \alpha$ |

575.

| | 0° . | 30° . | 45° . | 60° . | 90° . | 120° . | 135° . | 150° . | 180° . |
|-----|-------------|-----------------------|-----------------------|-----------------------|--------------|-----------------------|------------------------|------------------------|---------------|
| sin | 0 | $\frac{1}{2}$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}\sqrt{3}$ | 1 | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}$ | 0 |
| cos | 1 | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}\sqrt{2}$ | $-\frac{1}{2}\sqrt{3}$ | -1 |
| tan | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | ∞ | $-\sqrt{3}$ | -1 | $-\frac{1}{\sqrt{3}}$ | 0 |
| ctn | ∞ | $\sqrt{3}$ | 1 | $\frac{1}{\sqrt{3}}$ | 0 | $-\frac{1}{\sqrt{3}}$ | -1 | $-\sqrt{3}$ | ∞ |
| sec | 1 | $\frac{2}{\sqrt{3}}$ | $\sqrt{2}$ | 2 | ∞ | -2 | $-\sqrt{2}$ | $-\frac{2}{\sqrt{3}}$ | -1 |
| csc | ∞ | 2 | $\sqrt{2}$ | $\frac{2}{\sqrt{3}}$ | 1 | $\frac{2}{\sqrt{3}}$ | $\sqrt{2}$ | 2 | ∞ |

$$576. \sin \frac{1}{2}a = \sqrt{\frac{1}{2}(1 - \cos a)}.$$

$$577. \cos \frac{1}{2}a = \sqrt{\frac{1}{2}(1 + \cos a)}.$$

$$578. \tan \frac{1}{2}a = \sqrt{\frac{1 - \cos a}{1 + \cos a}} = \frac{1 - \cos a}{\sin a} = \frac{\sin a}{1 + \cos a}.$$

$$579. \sin 2a = 2 \sin a \cos a.$$

$$580. \sin 3a = 3 \sin a - 4 \sin^3 a.$$

$$581. \sin 4a = 8 \cos^3 a \cdot \sin a - 4 \cos a \sin a.$$

$$582. \sin 5a = 5 \sin a - 20 \sin^3 a + 16 \sin^5 a.$$

$$583. \sin 6a = 32 \cos^5 a \sin a - 32 \cos^3 a \sin a + 6 \cos a \sin a.$$

$$584. \cos 2a = \cos^2 a - \sin^2 a = 1 - 2 \sin^2 a = 2 \cos^2 a - 1.$$

$$585. \cos 3a = 4 \cos^3 a - 3 \cos a.$$

$$586. \cos 4a = 8 \cos^4 a - 8 \cos^2 a + 1.$$

$$587. \cos 5a = 16 \cos^5 a - 20 \cos^3 a + 5 \cos a.$$

$$588. \cos 6a = 32 \cos^6 a - 48 \cos^4 a + 18 \cos^2 a - 1.$$

$$589. \tan 2a = \frac{2 \tan a}{1 - \tan^2 a}.$$

$$590. \operatorname{ctn} 2a = \frac{\operatorname{ctn}^2 a - 1}{2 \operatorname{ctn} a}.$$

$$591. \sin(a \pm \beta) = \sin a \cdot \cos \beta \pm \cos a \cdot \sin \beta.$$

$$592. \cos(a \pm \beta) = \cos a \cdot \cos \beta \mp \sin a \cdot \sin \beta.$$

$$593. \tan(a \pm \beta) = \frac{\tan a \pm \tan \beta}{1 \mp \tan a \cdot \tan \beta}.$$

$$594. \operatorname{ctn}(a \pm \beta) = \frac{\operatorname{ctn} a \cdot \operatorname{ctn} \beta \mp 1}{\operatorname{ctn} a \pm \operatorname{ctn} \beta}.$$

$$595. \sin a \pm \sin \beta = 2 \sin \frac{1}{2}(a \pm \beta) \cdot \cos \frac{1}{2}(a \mp \beta).$$

$$596. \cos a + \cos \beta = 2 \cos \frac{1}{2}(a + \beta) \cdot \cos \frac{1}{2}(a - \beta).$$

$$597. \cos a - \cos \beta = -2 \sin \frac{1}{2}(a + \beta) \cdot \sin \frac{1}{2}(a - \beta).$$

$$598. \tan a \pm \tan \beta = \frac{\sin(a \pm \beta)}{\cos a \cdot \cos \beta}.$$

$$599. \operatorname{ctn} a \pm \operatorname{ctn} \beta = \pm \frac{\sin(a \pm \beta)}{\sin a \cdot \sin \beta}.$$

$$600. \frac{\sin a \pm \sin \beta}{\cos a + \cos \beta} = \tan \frac{1}{2}(a \pm \beta).$$

$$601. \frac{\sin a \pm \sin \beta}{\cos a - \cos \beta} = -\operatorname{ctn} \frac{1}{2}(a \mp \beta).$$

$$602. \frac{\sin a + \sin \beta}{\sin a - \sin \beta} = \frac{\tan \frac{1}{2}(a + \beta)}{\tan \frac{1}{2}(a - \beta)}.$$

$$603. \sin^2 a - \sin^2 \beta = \sin(a + \beta) \cdot \sin(a - \beta).$$

$$604. \cos^2 a - \cos^2 \beta = -\sin(a + \beta) \cdot \sin(a - \beta).$$

$$605. \cos^2 a - \sin^2 \beta = \cos(a + \beta) \cdot \cos(a - \beta).$$

$$606. \sin xi = \frac{1}{2}i(e^x - e^{-x}) = i \sinh x.$$

$$607. \cos xi = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$$

$$608. \tan xi = \frac{i(e^x - e^{-x})}{e^x + e^{-x}} = i \tanh x.$$

$$609. e^{x+yi} = e^x \cos y + ie^x \sin y.$$

$$610. a^{x+yi} = a^x \cos(y \cdot \log a) + ia^x \sin(y \cdot \log a).$$

$$611. (\cos \theta \pm i \cdot \sin \theta)^n = \cos n\theta \pm i \cdot \sin n\theta.$$

$$612. \sin x = -\frac{1}{2}i(e^{xi} - e^{-xi}).$$

$$613. \cos x = \frac{1}{2}(e^{xi} + e^{-xi}).$$

$$614. \tan x = -i \frac{e^{2xi} - 1}{e^{2xi} + 1}.$$

$$\begin{aligned} 615. \sin(x \pm yi) &= \sin x \cos yi \pm \cos x \sin yi \\ &= \sin x \cosh y \pm i \cos x \sinh y. \end{aligned}$$

$$\begin{aligned} 616. \cos(x \pm yi) &= \cos x \cos yi \mp \sin x \sin yi \\ &= \cos x \cosh y \mp i \sin x \sinh y. \end{aligned}$$

In any plane triangle,

$$617. \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$618. a^2 = b^2 + c^2 - 2bc \cos A.$$

$$619. \frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \frac{\operatorname{ctn} \frac{1}{2} C}{\tan \frac{1}{2}(A-B)}.$$

$$620. \sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ where } 2s = a + b + c.$$

$$621. \cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}.$$

$$622. \tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$623. \text{Area} = \frac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}.$$

In any spherical triangle,

$$624. \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

$$625. \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

$$626. -\cos A = \cos B \cos C - \sin B \sin C \cos a.$$

$$627. \sin a \operatorname{ctn} b = \sin C \operatorname{ctn} B + \cos a \cos C.$$

$$628. \cos \frac{1}{2} A = \sqrt{\frac{\sin s \cdot \sin(s-a)}{\sin b \cdot \sin c}}.$$

$$629. \sin \frac{1}{2} A = \sqrt{\frac{\sin(s-b) \cdot \sin(s-c)}{\sin b \cdot \sin c}}.$$

$$630. \tan \frac{1}{2} A = \sqrt{\frac{\sin(s-b) \cdot \sin(s-c)}{\sin s \cdot \sin(s-a)}}.$$

$$631. \cos \frac{1}{2} a = \sqrt{\frac{\cos(S-B) \cdot \cos(S-C)}{\sin B \cdot \sin C}}.$$

$$632. \sin \frac{1}{2} a = \sqrt{\frac{-\cos S \cdot \cos(S-A)}{\sin B \sin C}}.$$

$$633. \tan \frac{1}{2} a = \sqrt{\frac{-\cos S \cdot \cos(S-A)}{\cos(S-B) \cdot \cos(S-C)}}.$$

$$2s = a + b + c. \quad 2S = A + B + C.$$

$$634. \cos \frac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}c} \sin \frac{1}{2}C.$$

$$635. \cos \frac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}c} \sin \frac{1}{2}C.$$

$$636. \sin \frac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}c} \cos \frac{1}{2}C.$$

$$637. \sin \frac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}c} \cos \frac{1}{2}C.$$

$$638. \tan \frac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \operatorname{ctn} \frac{1}{2}C.$$

$$639. \tan \frac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \operatorname{ctn} \frac{1}{2}C.$$

$$640. \tan \frac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \frac{1}{2}c.$$

$$641. \tan \frac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \frac{1}{2}c.$$

$$642. \frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} = \frac{\operatorname{ctn} \frac{1}{2}C}{\tan \frac{1}{2}(A+B)}.$$

In interpreting equations which involve logarithmic and anti-trigonometric functions, it is necessary to remember that these functions are multiple valued. To save space the formulas on this page and the next are printed in contracted form.

$$\begin{aligned}
 643. \quad \sin^{-1} x &= \cos^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{x}{\sqrt{1-x^2}} = \sec^{-1} \frac{1}{\sqrt{1-x^2}} \\
 &= \csc^{-1} \frac{1}{x} = 2 \sin^{-1} \left[\frac{1}{2} - \frac{1}{2} \sqrt{1-x^2} \right]^{\frac{1}{2}} \\
 &= \frac{1}{2} \sin^{-1} (2x \sqrt{1-x^2}) = 2 \tan^{-1} \left[\frac{1 - \sqrt{1-x^2}}{x} \right] \\
 &= \frac{1}{2} \tan^{-1} \left[\frac{2x \sqrt{1-x^2}}{1-2x^2} \right] = \frac{1}{2} \pi - \cos^{-1} x \\
 &= \frac{1}{2} \pi - \sin^{-1} \sqrt{1-x^2} = -\sin^{-1}(-x) \\
 &= \operatorname{ctn}^{-1} \frac{\sqrt{1-x^2}}{x} = (2n + \frac{1}{2})\pi - i \log(x + \sqrt{x^2-1}) \\
 &= \frac{1}{4} \pi + \frac{1}{2} \sin^{-1}(2x^2-1) = \frac{1}{2} \cos^{-1}(1-2x^2).
 \end{aligned}$$

$$\begin{aligned}
 644. \quad \cos^{-1} x &= \sin^{-1} \sqrt{1-x^2} = \tan^{-1} \frac{\sqrt{1-x^2}}{x} = \sec^{-1} \frac{1}{x} \\
 &= \frac{1}{2} \pi - \sin^{-1} x = 2 \cos^{-1} \sqrt{\frac{1+x}{2}} \\
 &= \frac{1}{2} \cos^{-1}(2x^2-1) \\
 &= 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} = \frac{1}{2} \tan^{-1} \left[\frac{2x \sqrt{1-x^2}}{2x^2-1} \right] \\
 &= \csc^{-1} \frac{1}{\sqrt{1-x^2}} = \pi - \cos^{-1}(-x) \\
 &= \operatorname{ctn}^{-1} \frac{x}{\sqrt{1-x^2}} \\
 &= i \log(x + \sqrt{x^2-1}) = \pi - i \log(\sqrt{x^2-1} - x).
 \end{aligned}$$

$$\begin{aligned}
645. \quad \tan^{-1} x &= \sin^{-1} \frac{x}{\sqrt{1+x^2}} = \cos^{-1} \frac{1}{\sqrt{1+x^2}} = \frac{1}{2} \sin^{-1} \frac{2x}{1+x^2} \\
&= \operatorname{ctn}^{-1} \frac{1}{x} = \frac{1}{2} \pi - \operatorname{ctn}^{-1} x = \sec^{-1} \sqrt{1+x^2} \\
&= \frac{1}{2} \pi - \tan^{-1} \frac{1}{x} \\
&= \csc^{-1} \frac{\sqrt{1+x^2}}{x} = \frac{1}{2} \cos^{-1} \left[\frac{1-x^2}{1+x^2} \right] \\
&= 2 \cos^{-1} \left[\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}} \right]^{\frac{1}{2}} = 2 \sin^{-1} \left[\frac{\sqrt{1+x^2}-1}{2\sqrt{1+x^2}} \right]^{\frac{1}{2}} \\
&= \frac{1}{2} \tan^{-1} \frac{2x}{1-x^2} = 2 \tan^{-1} \left[\frac{\sqrt{1+x^2}-1}{x} \right] \\
&= -\tan^{-1} c + \tan^{-1} \left[\frac{x+c}{1-cx} \right] = -\tan^{-1}(-x) \\
&= \frac{1}{2} i \log \frac{1-xi}{1+xi} = \frac{1}{2} i \log \frac{i+x}{i-x} \\
&= -\frac{1}{2} i \log \frac{1+xi}{1-xi}.
\end{aligned}$$

$$646. \quad \sin^{-1} x \pm \sin^{-1} y = \sin^{-1} [x \sqrt{1-y^2} \pm y \sqrt{1-x^2}].$$

$$647. \quad \cos^{-1} x \pm \cos^{-1} y = \cos^{-1} [xy \mp \sqrt{(1-x^2)(1-y^2)}].$$

$$648. \quad \tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left[\frac{x \pm y}{1 \mp xy} \right].$$

$$\begin{aligned}
649. \quad \sin^{-1} x \pm \cos^{-1} y &= \sin^{-1} [xy \pm \sqrt{(1-x^2)(1-y^2)}] \\
&= \cos^{-1} [y \sqrt{1-x^2} \mp x \sqrt{1-y^2}].
\end{aligned}$$

$$650. \quad \tan^{-1} x \pm \operatorname{ctn}^{-1} y = \tan^{-1} \left[\frac{xy \pm 1}{y \mp x} \right] = \operatorname{ctn}^{-1} \left[\frac{y \mp x}{xy \pm 1} \right]$$

$$651. \quad \log (x + yi) = \frac{1}{2} \log (x^2 + y^2) + i \tan^{-1} (y/x).$$

B. — HYPERBOLIC FUNCTIONS.

$$652. \sinh x = \frac{1}{2}(e^x - e^{-x}) = -\sinh(-x) = -i \sin(ix) \\ = (\operatorname{csch} x)^{-1} = 2 \tanh \frac{1}{2} x \div (1 - \tanh^2 \frac{1}{2} x).$$

$$653. \cosh x = \frac{1}{2}(e^x + e^{-x}) = \cosh(-x) = \cos(ix) = (\operatorname{sech} x)^{-1} \\ = (1 + \tanh^2 \frac{1}{2} x) \div (1 - \tanh^2 \frac{1}{2} x).$$

$$654. \tanh x = (e^x - e^{-x}) \div (e^x + e^{-x}) = -\tanh(-x) \\ = -i \tan(ix) = (\operatorname{ctnh} x)^{-1} = \sinh x \div \cosh x.$$

$$655. \cosh xi = \cos x. \quad \cos ix = \cosh x$$

$$656. \sinh xi = i \sin x. \quad \Rightarrow \sin ix = i \sinh x$$

$$657. \cosh^2 x - \sinh^2 x = 1.$$

$$658. 1 - \tanh^2 x = \operatorname{sech}^2 x.$$

$$659. 1 - \operatorname{ctnh}^2 x = -\operatorname{csch}^2 x.$$

$$660. \sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y.$$

$$661. \cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y.$$

$$662. \tanh(x \pm y) = (\tanh x \pm \tanh y) \div (1 \pm \tanh x \cdot \tanh y).$$

$$663. \sinh(2x) = 2 \sinh x \cosh x.$$

$$664. \cosh(2x) = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x.$$

$$665. \tanh(2x) = 2 \tanh x \div (1 + \tanh^2 x).$$

$$666. \sinh(\frac{1}{2} x) = \sqrt{\frac{1}{2} (\cosh x - 1)}.$$

$$667. \cosh(\frac{1}{2} x) = \sqrt{\frac{1}{2} (\cosh x + 1)}.$$

$$668. \tanh(\frac{1}{2} x) = (\cosh x - 1) \div \sinh x = \sinh x \div (\cosh x + 1).$$

$$669. \sinh x + \sinh y = 2 \sinh \frac{1}{2} (x + y) \cdot \cosh \frac{1}{2} (x - y).$$

$$670. \sinh x - \sinh y = 2 \cosh \frac{1}{2} (x + y) \cdot \sinh \frac{1}{2} (x - y).$$

$$671. \cosh x + \cosh y = 2 \cosh \frac{1}{2}(x+y) \cdot \cosh \frac{1}{2}(x-y).$$

$$672. \cosh x - \cosh y = 2 \sinh \frac{1}{2}(x+y) \cdot \sinh \frac{1}{2}(x-y)$$

$$673. d \sinh x = \cosh x \cdot dx.$$

$$674. d \cosh x = \sinh x \cdot dx.$$

$$675. d \tanh x = \operatorname{sech}^2 x \cdot dx.$$

$$676. d \operatorname{ctnh} x = -\operatorname{csch}^2 x \cdot dx.$$

$$677. d \operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x \cdot dx.$$

$$678. d \operatorname{csch} x = -\operatorname{csch} x \cdot \operatorname{ctnh} x \cdot dx.$$

$$679. \sinh^{-1} x = \log(x + \sqrt{x^2 + 1}) = \int \frac{dx}{\sqrt{x^2 + 1}} \\ = \cosh^{-1} \sqrt{x^2 + 1}.$$

$$680. \cosh^{-1} x = \log(x + \sqrt{x^2 - 1}) = \int \frac{dx}{\sqrt{x^2 - 1}} \\ = \sinh^{-1} \sqrt{x^2 - 1}.$$

$$681. \tanh^{-1} x = \frac{1}{2} \log(1+x) - \frac{1}{2} \log(1-x) = \int \frac{dx}{1-x^2}.$$

$$682. \operatorname{ctnh}^{-1} x = \frac{1}{2} \log(1+x) - \frac{1}{2} \log(x-1) = \int \frac{dx}{1-x^2}$$

$$683. \operatorname{sech}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) = -\int \frac{dx}{x\sqrt{1-x^2}}.$$

$$684. \operatorname{csch}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right) = -\int \frac{dx}{x\sqrt{x^2 + 1}}.$$

$$685. d \sinh^{-1} x = \frac{dx}{\sqrt{1+x^2}}.$$

$$686. d \cosh^{-1} x = \frac{dx}{\sqrt{x^2 - 1}}.$$

$$687. \quad d \tanh^{-1} x = \frac{dx}{1-x^2}.$$

$$688. \quad d \operatorname{ctnh}^{-1} x = -\frac{dx}{x^2-1}.$$

$$689. \quad d \operatorname{sech}^{-1} x = -\frac{dx}{x\sqrt{1-x^2}}.$$

$$690. \quad d \operatorname{csch}^{-1} x = -\frac{dx}{x\sqrt{x^2+1}}.$$

If m is an integer,

$$691. \quad \sinh(m\pi i) = 0.$$

$$692. \quad \cosh(m\pi i) = \cos m\pi = (-1)^m.$$

$$693. \quad \tanh(m\pi i) = 0.$$

$$694. \quad \sinh(x + m\pi i) = (-1)^m \sinh x.$$

$$695. \quad \cosh(x + m\pi i) = (-1)^m \cosh(x).$$

$$696. \quad \sinh(2m+1)\frac{1}{2}\pi i = i \sin(2m+1)\frac{1}{2}\pi = \pm i$$

$$697. \quad \cosh(2m+1)\frac{1}{2}\pi i = 0.$$

$$698. \quad \sinh\left(\frac{\pi i}{2} \pm x\right) = i \cosh x.$$

$$799. \quad \cosh\left(\frac{\pi i}{2} \pm x\right) = \pm i \sinh x.$$

$$700. \quad \sinh u = \tan \operatorname{gd} u.$$

$$701. \quad \cosh u = \sec \operatorname{gd} u.$$

$$702. \quad \tanh u = \sin \operatorname{gd} u.$$

$$703. \quad \tanh \frac{1}{2} u = \tan \frac{1}{2} \operatorname{gd} u.$$

$$704. \quad u = \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}\operatorname{gd} u\right). \quad \int \sec x \, dx = \operatorname{gd}^{-1} x.$$

C. — ELLIPTIC FUNCTIONS.

$$\text{If } u \equiv F(\phi, k) \equiv \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \equiv \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

where $k < 1$, and $x \equiv \sin \phi$, ϕ is called the *amplitude* of u and is written $\text{am}(u, \text{mod } k)$, or, more simply, $\text{am } u$; $x \equiv \sin \phi \equiv \text{sn } u$,

$$\sqrt{1-x^2} \equiv \cos \phi \equiv \text{cn } u, \quad \sqrt{1-k^2x^2} \equiv \Delta \phi \equiv \Delta \text{n } u \equiv \text{dn } u,$$

$$K \equiv F(\tfrac{1}{2}\pi, k), \quad K' \equiv F(\tfrac{1}{2}\pi, k').$$

$$\text{Hence, } \text{am}(0) = 0, \quad \text{sn}(0) = 0, \quad \text{cn}(0) = 1, \quad \text{dn}(0) = 1,$$

$$\text{am}(-u) = -\text{am } u, \quad \text{sn}(-u) = -\text{sn } u,$$

$$\text{cn}(-u) = \text{cn } u, \quad \text{dn}(-u) = \text{dn } u.$$

$$705. \quad \text{sn}^2 u + \text{cn}^2 u = 1.$$

$$706. \quad \text{dn}^2 u + k^2 \text{sn}^2 u = 1.$$

$$707. \quad \text{dn}^2 u - k^2 \text{cn}^2 u = 1 - k^2 = k'^2.$$

$$708. \quad \text{sn } 2u = \frac{2 \text{sn } u \cdot \text{cn } u \cdot \text{dn } u}{1 - k^2 \text{sn}^4 u}.$$

$$\begin{aligned} 709. \quad \text{cn } 2u &= \frac{\text{cn}^2 u - \text{sn}^2 u \cdot \text{dn}^2 u}{1 - k^2 \text{sn}^4 u} = \frac{1 - 2 \text{sn}^2 u + k^2 \text{sn}^4 u}{1 - k^2 \text{sn}^4 u} \\ &= 1 - \frac{2 \text{sn}^2 u \cdot \text{dn}^2 u}{1 - k^2 \text{sn}^4 u} = \frac{2 \text{cn}^2 u}{1 - k^2 \text{sn}^4 u} - 1. \end{aligned}$$

$$\begin{aligned} 710. \quad \text{dn } 2u &= \frac{\text{dn}^2 u - k^2 \text{sn}^2 u \cdot \text{cn}^2 u}{1 - k^2 \text{sn}^4 u} = \frac{1 - 2 k^2 \text{sn}^2 u + k^2 \text{sn}^4 u}{1 - k^2 \text{sn}^4 u} \\ &= 1 - \frac{2 k^2 \text{sn}^2 u \cdot \text{cn}^2 u}{1 - k^2 \text{sn}^4 u} = \frac{2 \text{dn}^2 u}{1 - k^2 \text{sn}^4 u} - 1. \end{aligned}$$

$$711. \quad \text{sn}^2\left(\frac{u}{2}\right) = \frac{1 - \text{cn } u}{1 + \text{dn } u} = \frac{1 - \text{dn } u}{k^2(1 + \text{cn } u)} = \frac{\text{dn } u - \text{cn } u}{k'^2 + \text{dn } u - k^2 \text{cn } u}.$$

$$\begin{aligned} 712. \quad \text{cn}^2\left(\frac{u}{2}\right) &= \frac{\text{dn } u + \text{cn } u}{1 + \text{dn } u} = \frac{k^2 \text{cn } u - k'^2 + \text{dn } u}{k^2(1 + \text{cn } u)} \\ &= \frac{k'^2(1 + \text{cn } u)}{k'^2 + \text{dn } u - k^2 \text{cn } u}. \end{aligned}$$

$$\begin{aligned}
 713. \quad \operatorname{dn}^2\left(\frac{u}{2}\right) &= \frac{k'^2 + \operatorname{dn} u + k^2 \operatorname{cn} u}{1 + \operatorname{dn} u} = \frac{k^2 (\operatorname{cn} u + \operatorname{dn} u)}{k^2 (1 + \operatorname{cn} u)} \\
 &= \frac{k'^2 (1 + \operatorname{dn} u)}{k'^2 + \operatorname{dn} u - k^2 \operatorname{cn} u}.
 \end{aligned}$$

$$\text{If, moreover, } v = \int_0^y \frac{dz}{\sqrt{(1-z^2)(1-k^2 z^2)}},$$

$$714. \quad \operatorname{sn}^2 u - \operatorname{sn}^2 v = \operatorname{cn}^2 v - \operatorname{cn}^2 u.$$

$$715. \quad \operatorname{sn}(u \pm v) = \frac{\operatorname{sn} u \cdot \operatorname{cn} v \cdot \operatorname{dn} v \pm \operatorname{cn} u \cdot \operatorname{sn} v \cdot \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$\begin{aligned}
 716. \quad \operatorname{cn}(u \pm v) &= \frac{\operatorname{cn} u \cdot \operatorname{cn} v \mp \operatorname{sn} u \cdot \operatorname{sn} v \cdot \operatorname{dn} u \cdot \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} \\
 &= \operatorname{cn} u \cdot \operatorname{cn} v \mp \operatorname{sn} u \cdot \operatorname{sn} v \cdot \operatorname{dn}(u \pm v).
 \end{aligned}$$

$$\begin{aligned}
 717. \quad \operatorname{dn}(u \pm v) &= \frac{\operatorname{dn} u \cdot \operatorname{dn} v \mp k^2 \operatorname{sn} u \cdot \operatorname{sn} v \cdot \operatorname{cn} u \cdot \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} \\
 &= \operatorname{dn} u \cdot \operatorname{dn} v \mp k^2 \operatorname{sn} u \cdot \operatorname{sn} v \cdot \operatorname{cn}(u \pm v).
 \end{aligned}$$

$$718. \quad \operatorname{tn}(u \pm v) = \frac{\operatorname{tn} u \cdot \operatorname{dn} v \pm \operatorname{tn} v \cdot \operatorname{dn} u}{1 \mp \operatorname{tn} u \cdot \operatorname{tn} v \cdot \operatorname{dn} u \cdot \operatorname{dn} v}.$$

$$719. \quad \operatorname{sn}(u + v) + \operatorname{sn}(u - v) = \frac{2 \operatorname{sn} u \cdot \operatorname{cn} v \cdot \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$720. \quad \operatorname{sn}(u + v) - \operatorname{sn}(u - v) = \frac{2 \operatorname{sn} v \cdot \operatorname{cn} u \cdot \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$721. \quad \operatorname{cn}(u + v) + \operatorname{cn}(u - v) = \frac{2 \operatorname{cn} u \cdot \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$722. \quad \operatorname{cn}(u + v) - \operatorname{cn}(u - v) = -\frac{2 \operatorname{sn} u \cdot \operatorname{sn} v \cdot \operatorname{dn} u \cdot \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$723. \quad \operatorname{dn}(u + v) + \operatorname{dn}(u - v) = \frac{2 \operatorname{dn} u \cdot \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$724. \quad \operatorname{dn}(u+v) - \operatorname{dn}(u-v) = -\frac{2k^2 \operatorname{sn} u \cdot \operatorname{sn} v \cdot \operatorname{cn} u \cdot \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$725. \quad \operatorname{sn}(u+v) \cdot \operatorname{sn}(u-v) = \frac{\operatorname{sn}^2 u - \operatorname{sn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}$$

$$= \frac{\operatorname{cn}^2 v + \operatorname{sn}^2 u \cdot \operatorname{dn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} - 1 = \frac{1}{k^2} \left[\frac{\operatorname{dn}^2 v + k^2 \operatorname{sn}^2 u \cdot \operatorname{cn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} - 1 \right].$$

$$726. \quad \operatorname{cn}(u+v) \cdot \operatorname{cn}(u-v) = \frac{\operatorname{cn}^2 u - \operatorname{sn}^2 v + k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}$$

$$= \frac{\operatorname{cn}^2 u + \operatorname{cn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} - 1 = 1 - \frac{\operatorname{sn}^2 u \cdot \operatorname{dn}^2 v + \operatorname{sn}^2 v \cdot \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$727. \quad \operatorname{dn}(u+v) \cdot \operatorname{dn}(u-v)$$

$$= \frac{1 - k^2 \operatorname{sn}^2 u - k^2 \operatorname{sn}^2 v + k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}$$

$$= \frac{\operatorname{dn}^2 u + \operatorname{dn}^2 v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v} - 1.$$

$$728. \quad \operatorname{sn}(u \pm v) \operatorname{cn}(u \mp v) = \frac{\operatorname{sn} u \cdot \operatorname{cn} u \cdot \operatorname{dn} v \pm \operatorname{sn} v \cdot \operatorname{cn} v \cdot \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$729. \quad \operatorname{sn}(u \pm v) \operatorname{dn}(u \mp v) = \frac{\operatorname{sn} u \cdot \operatorname{dn} u \cdot \operatorname{cn} v \pm \operatorname{sn} v \cdot \operatorname{dn} v \cdot \operatorname{cn} u}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$730. \quad \operatorname{cn}(u \pm v) \operatorname{dn}(u \mp v) = \frac{\operatorname{cn} u \cdot \operatorname{dn} u \cdot \operatorname{cn} v \cdot \operatorname{dn} v \mp k'^2 \operatorname{sn} u \cdot \operatorname{sn} v}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$731. \quad [1 \pm \operatorname{sn}(u+v)][1 \pm \operatorname{sn}(u-v)] = \frac{(\operatorname{cn} v \pm \operatorname{sn} u \cdot \operatorname{dn} v)^2}{1 - k^2 \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}.$$

$$732. \quad \operatorname{sn}(ui, k) = i \operatorname{sn}(u, k') / \operatorname{cn}(u, k').$$

$$733. \quad \operatorname{cn}(ui, k) = 1 / \operatorname{cn}(u, k').$$

$$734. \quad \operatorname{dn}(ui, k) = \operatorname{dn}(u, k') / \operatorname{cn}(u, k').$$

D. — BESSEL'S FUNCTIONS.

$$735. J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$$736. K_0(x) = J_0(x) \cdot \log x + \frac{x^2}{2^2} - \frac{x^4 \cdot \Omega_2}{2^2 \cdot 4^2} + \frac{x^6 \cdot \Omega_3}{2^2 \cdot 4^2 \cdot 6^2} - \dots$$

$$[\Omega_k = 1 + \frac{1}{2} + \frac{1}{3} + \dots + 1/k.]$$

$$737. J_n(x) = \frac{n!}{\Gamma(n+1)} \sum_0^{\infty} \frac{(-1)^k x^{n+2k}}{2^{n+2k} \cdot k! (n+k)!} \quad [\text{When } n \text{ is an integer} \\ \text{819 may be used.}]$$

$$738. K_n(x) = J_n(x) \cdot \log x - \frac{x^{-n}}{2^{1-n}} \sum_0^{n-1} \frac{(n-k-1)! x^{2k}}{2^{2k} \cdot k!} \\ - \frac{x^n}{2^{1+n}} \sum_0^{\infty} \frac{(-1)^k}{(n+k)! k!} \left[\Omega_k + \Omega_{k+n} \left(\frac{x}{2} \right)^{2k} \right].$$

739. According as n is or is not an integer, $A \cdot J_n(x) + B \cdot K_n(x)$, or $A \cdot J_n(x) + B \cdot J_{-n}(x)$ is a particular solution of Bessel's equation,

$$\frac{d^2 z}{dx^2} + \frac{1}{x} \cdot \frac{dz}{dx} + \left(1 - \frac{n^2}{x^2} \right) z = 0.$$

$$740. dJ_0(x)/dx = -J_1(x); d[x^n \cdot J_n(x)]/dx = x^n \cdot J_{n-1}(x), \\ \text{if } n > \frac{1}{2}; d[x^{-n} \cdot J_n(x)]/dx = -x^{-n} \cdot J_{n+1}(x), \text{ if } n > -\frac{1}{2}.$$

$$741. J_{n-1}(x) - J_{n+1}(x) = 2 \cdot dJ_n(x)/dx; \\ 2n \cdot J_n(x) = x \cdot J_{n-1}(x) + x \cdot J_{n+1}(x).$$

When x is large it is sometimes convenient to compute approximate numerical values of $J_n(x)$ by means of the semi-convergent series,

$$742. J_n(x) = \sqrt{\frac{2}{\pi x}} \left[P_n \cdot \cos \left\{ \frac{(2n+1)\pi}{4} - x \right\} \right. \\ \left. + Q_n \cdot \sin \left\{ \frac{(2n+1)\pi}{4} - x \right\} \right].$$

$$743. P_n = 1 - \frac{(4n^2-1)(4n^2-9)}{2!(8x)^2} \\ + \frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{4!(8x)^4} - \dots$$

$$744. Q_n = \frac{4n^2-1}{8x} - \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8x)^3} + \dots$$

E. — SERIES AND PRODUCTS.

[The expression in brackets attached to an infinite series shows values of the variable which lie within the interval of convergence. If a series is convergent for all finite values of x , the expression $[x^2 < \infty]$ is used.]

$$745. (a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \cdots + \frac{n! a^{n-k}b^k}{(n-k)! k!} + \cdots. [b^2 < a^2.]$$

$$746. (a - bx)^{-1} = \frac{1}{a} \left[1 + \frac{bx}{a} + \frac{b^2x^2}{a^2} + \frac{b^3x^3}{a^3} + \cdots \right]. [b^2x^2 < a^2.]$$

$$747. (1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!} x^2 \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots + \frac{(\pm 1)^k n! x^k}{(n-k)! k!} + \cdots. [x^2 < 1.]$$

$$748. (1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!} x^2 \mp \frac{n(n+1)(n+2)x^3}{3!} + \cdots (\mp)^k \frac{(n+k-1)! x^k}{(n-1)! k!} + \cdots. [x^2 < 1.]$$

$$749. (1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1 \cdot 1}{2 \cdot 4} x^2 \pm \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} x^4 \pm \cdots. [x^2 < 1.]$$

$$750. (1 \pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4} x^2 \mp \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} x^4 \mp \cdots. [x^2 < 1.]$$

$$751. (1 \pm x)^{\frac{3}{2}} = 1 \pm \frac{3}{2}x - \frac{1 \cdot 2}{3 \cdot 6} x^2 \pm \frac{1 \cdot 2 \cdot 5}{3 \cdot 6 \cdot 9} x^3 - \frac{1 \cdot 2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} x^4 \pm \cdots. [x^2 < 1.]$$

$$752. (1 \pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{1 \cdot 4}{3 \cdot 6}x^2 \mp \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9}x^3 \\ + \frac{1 \cdot 4 \cdot 7 \cdot 10}{3 \cdot 6 \cdot 9 \cdot 12}x^4 \mp \dots \quad [x^2 < 1.]$$

$$753. (1 \pm x^2)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x^2 - \frac{x^4}{2 \cdot 4} \pm \frac{1 \cdot 3 \cdot x^6}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5 \cdot x^8}{2 \cdot 4 \cdot 6 \cdot 8} \pm \dots \\ [x^2 < 1.]$$

$$754. (1 \pm x^2)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 \mp \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \dots \\ [x^2 < 1.]$$

$$755. (1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \dots \quad [x^2 < 1.]$$

$$756. (1 \pm x)^{\frac{3}{2}} = 1 \pm \frac{3}{2}x + \frac{3 \cdot 1}{2 \cdot 4}x^2 \mp \frac{3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6}x^3 \\ + \frac{3 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 8}x^4 \mp \frac{3 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}x^5 + \dots \quad [x^2 < 1.]$$

$$757. (1 \pm x)^{-\frac{3}{2}} = 1 \mp \frac{3}{2}x + \frac{3 \cdot 5}{2 \cdot 4}x^2 \mp \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6}x^3 + \dots \quad [x^2 < 1.]$$

$$758. (1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \dots \\ [x^2 < 1.]$$

$$759. e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad [x^2 < \infty.]$$

$$760. a^x = 1 + x \log a + \frac{(x \log a)^2}{2!} + \frac{(x \log a)^3}{3!} + \dots \quad [x^2 < \infty.]$$

$$761. \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \quad [x^2 < \infty.]$$

$$762. \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \quad [x^2 < \infty.]$$

$$763. e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots \quad [x^2 < \infty.]$$

A series of numbers, $B_1, B_2, B_3 \dots$, of odd and even orders, which appear in the developments of many functions, may be computed by means of the equations,

$$B_{2n} - \frac{2n(2n-1)}{2!} B_{2n-2} + \frac{2n(2n-1)(2n-2)(2n-3)}{4!} B_{2n-4} - \dots (-1)^n = 0.$$

$$\frac{2^{2n}(2^{2n}-1)}{2n} B_{2n-1} = (2n-1) B_{2n-2} - \frac{(2n-1)(2n-2)(2n-3)}{3!} B_{2n-4} + \dots (-1)^{n-1} = 0.$$

Whence $B_1 = \frac{1}{6}$, $B_2 = 1$, $B_3 = \frac{1}{30}$, $B_4 = 5$, $B_5 = \frac{1}{42}$, $B_6 = 61$, $B_7 = \frac{1}{30}$, $B_8 = 1385$, $B_9 = \frac{5}{66}$, $B_{10} = 50521$, $B_{11} = \frac{691}{2730}$, $B_{12} = 2702765$, $B_{13} = \frac{7}{6}$, etc. The B 's of odd orders are called Bernoulli's Numbers; those of even orders, Euler's Numbers. What are here denoted by B_{2n-1} and B_{2n} are sometimes represented by B_n and E_n , respectively,

$$\frac{B_{2n-1}}{(2n)!} = \frac{2}{(2^{2n}-1)\pi^{2n}} \left[1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \dots \right],$$

$$\frac{B_{2n}}{(2n)!} = \frac{2^{2n+2}}{\pi^{2n+1}} \left[1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \dots \right].$$

$$764. \quad \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_3 x^4}{4!} + \frac{B_5 x^6}{6!} - \frac{B_7 x^8}{8!} + \dots$$

[$x < 2\pi$.]

$$765. \quad \log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$$

[$2 > x > 0$.]

$$766. \quad \log x = \frac{x-1}{x} + \frac{1}{2} \left(\frac{x-1}{x} \right)^2 + \frac{1}{3} \left(\frac{x-1}{x} \right)^3 + \dots$$

[$x > \frac{1}{2}$.]

$$767. \log x = 2 \left[\frac{x-1}{x+1} + \frac{1}{3} \left(\frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left(\frac{x-1}{x+1} \right)^5 + \dots \right]. \quad [x > 0.]$$

$$768. \log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots. \quad [x^2 < 1.]$$

$$769. \log \left(\frac{1+x}{1-x} \right) = 2 \left[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots \right]. \quad [x^2 < 1.]$$

$$770. \log \left(\frac{x+1}{x-1} \right) = 2 \left[\frac{1}{x} + \frac{1}{3} \left(\frac{1}{x} \right)^3 + \frac{1}{5} \left(\frac{1}{x} \right)^5 + \dots \right]. \quad [x^2 > 1.]$$

$$771. \log(x + \sqrt{1+x^2}) = x - \frac{1}{6}x^3 + \frac{1 \cdot 3}{2 \cdot 4}x^5 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^7 + \dots. \quad [x^2 < 1.]$$

Series for denary and other logarithms can be obtained from the foregoing developments by aid of the equations,

$$\log_a x = \log_e x \cdot \log_a e, \quad \log_e x = \log_a x \cdot \log_e a,$$

$$\log_e(-z) = (2n+1)\pi i + \log_e z.$$

$$772. \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots. \quad [x^2 < \infty.]$$

$$773. \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = 1 - \text{versin } x. \quad [x^2 < \infty.]$$

$$774. \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots + \frac{2^{2n}(2^{2n}-1)B_{2n-1}x^{2n-1}}{(2n)!} + \dots. \quad [x^2 < \frac{1}{4}\pi^2.]$$

$$775. \text{ctn } x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \dots - \frac{B_{2n-1}(2x)^{2n}}{x(2n)!} - \dots. \quad [x^2 < \pi^2.]$$

$$776. \sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots + \frac{B_{2n}x^{2n}}{(2n)!} + \cdots, \quad \left[x^2 < \frac{\pi^2}{4} \right]$$

$$777. \csc x = \frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3 \cdot 5!} + \frac{31x^5}{3 \cdot 7!} \\ + \cdots + \frac{2(2^{2n+1}-1)}{(2n+2)!} B_{2n+1} x^{2n+1} + \cdots, \quad [x^2 < \pi^2.]$$

$$778. \sin^{-1} x = x + \frac{x^3}{6} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} \\ + \cdots = \frac{1}{2} \pi - \cos^{-1} x, \quad [x^2 < 1.]$$

$$779. \tan^{-1} x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \cdots = \frac{1}{2} \pi - \cot^{-1} x, \\ [x^2 < 1.]$$

$$780. \tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, \quad [x^2 > 1.]$$

$$781. \sec^{-1} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{6x^3} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7x^7} - \cdots \\ = \frac{1}{2} \pi - \csc^{-1} x, \quad [x^2 > 1.]$$

$$782. \log \sin x = \log x - \frac{1}{6} x^2 - \frac{1}{180} x^4 - \frac{1}{835} x^6 \\ - \cdots - \frac{2^{2n-1} B_{2n-1} x^{2n}}{n(2n)!} - \cdots, \quad [x^2 < \pi^2.]$$

$$783. \log \cos x = -\frac{1}{2} x^2 - \frac{1}{12} x^4 - \frac{1}{45} x^6 - \frac{1}{2520} x^8 \\ - \cdots - \frac{2^{2n-1} (2^{2n}-1) B_{2n-1} x^{2n}}{n(2n)!} - \cdots, \quad [x^2 < \frac{1}{4} \pi^2.]$$

$$784. \log \tan x = \log x + \frac{1}{8} x^2 + \frac{7}{960} x^4 + \frac{63}{2835} x^6 \\ + \cdots + \frac{(2^{2n-1}-1) 2^{2n} B_{2n-1} x^{2n}}{n(2n)!} + \cdots, \quad [x^2 < \frac{1}{4} \pi^2.]$$

$$785. e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^6}{5!} - \frac{3x^8}{6!} + \frac{56x^7}{7!} + \cdots, \\ [x^2 < \infty.]$$

$$786. e^{\cos x} = e \left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots \right). \quad [x^2 < \infty.]$$

$$787. e^{\tan x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \cdots. \quad [x^2 < \frac{1}{4}\pi^2.]$$

$$788. e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \cdots. \quad [x^2 < 1.]$$

$$789. e^{\tan^{-1} x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} - \cdots. \quad [x^2 < 1.]$$

$$790. \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots. \quad [x^2 < \infty.]$$

$$791. \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots. \quad [x^2 < \infty.]$$

$$792. \tanh x = (2^2 - 1) 2^2 B_1 \frac{x}{2!} - (2^4 - 1) 2^4 B_3 \frac{x^3}{4!} + \cdots$$

$$= \Sigma [(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n-1} x^{2n-1} / (2n)!].$$

$$[x^2 < \frac{1}{4}\pi^2.]$$

$$793. \operatorname{ctnh} x = \frac{1}{x} (1 + \Sigma [(-1)^{n-1} 2^{2n} B_{2n-1} x^{2n} / (2n)!]).$$

$$[x^2 < \pi^2.]$$

$$794. \operatorname{sech} x = 1 + \Sigma [(-1)^n B_{2n} x^{2n} / (2n)!]. \quad [x^2 < \frac{1}{4}\pi^2.]$$

$$795. \operatorname{csch} x = \frac{1}{x} - (2 - 1) 2 B_1 \frac{x}{2!} + (2^3 - 1) 2 B_3 \frac{x^3}{4!} - \cdots$$

$$= \frac{1}{x} (1 + 2 \Sigma [(-1)^n (2^{2n-1} - 1) B_{2n-1} x^{2n} / (2n)!]).$$

$$[x^2 < \pi^2.]$$

$$796. \sinh^{-1} x = x - \frac{1}{6} x^3 + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5 \cdot x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots. \quad [x^2 < 1.]$$

$$797. \tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots. \quad [x^2 < 1.]$$

$$798. \operatorname{etnh}^{-1} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots. \quad [x^2 > 1.]$$

$$799. \operatorname{csch}^{-1} x = \frac{1}{x} - \frac{1}{2 \cdot 3 \cdot x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot x^5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot x^7} + \cdots. \quad [x^2 > 1.]$$

$$800. \int_0^x e^{-x^2} dx = x - \frac{1}{3} x^3 + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots. \quad [x^2 < \infty.]$$

$$801. \int_0^x \cos(x^2) dx = x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \cdots. \quad [x^2 < \infty.]$$

$$802. \int_0^1 \frac{x^{a-1} dx}{1+x^b} = \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \cdots.$$

$$803. f(x+h) = f(x) + h \cdot f'(x + \theta h).$$

$$804. f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} \cdot f^n(x + \theta h).$$

$$805. f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{(n-1)!} \cdot (1-\theta)^{n-1} \cdot f^n(x + \theta h).$$

$$806. f(x+h, y+k) = f(x, y) + hf'_x(x + \theta h, y + \theta k) + kf'_y(x + \theta h, y + \theta k).$$

$$807. f(x+h, y+k) = f(x, y) + \left(h \frac{\partial f(x, y)}{\partial x} + k \frac{\partial f(x, y)}{\partial y} \right) + \frac{1}{2!} \left(h^2 \frac{\partial^2 f(x, y)}{\partial x^2} + 2hk \frac{\partial^2 f(x, y)}{\partial x \cdot \partial y} + k^2 \frac{\partial^2 f(x, y)}{\partial y^2} \right)$$

$$\begin{aligned}
 & + \frac{1}{3!} \left(h^3 \frac{\partial^3 f(x, y)}{\partial x^3} + 3 h^2 k \frac{\partial^3 f(x, y)}{\partial y \cdot \partial x^2} + 3 h k^2 \frac{\partial^3 f(x, y)}{\partial x \cdot \partial y^2} \right. \\
 & \quad \left. + k^3 \frac{\partial^3 f(x, y)}{\partial y^3} \right) + \dots + R_n \\
 & = f(x, y) + (hD_x + kD_y)f(x, y) + \frac{1}{2!} (hD_x + kD_y)^2 f(x, y) \\
 & \quad + \dots + \frac{1}{(n-1)!} (hD_x + kD_y)^{n-1} f(x, y) \\
 & \quad + \frac{1}{n!} (hD_x + kD_y)^n f(x + \theta h, y + \theta k).
 \end{aligned}$$

$$808. \quad 1 = \frac{4}{\pi} \left[\sin \frac{\pi x}{c} + \frac{1}{3} \sin \frac{3\pi x}{c} + \frac{1}{5} \sin \frac{5\pi x}{c} + \dots \right].$$

[0 < x < c.]

$$809. \quad x = \frac{2c}{\pi} \left[\sin \frac{\pi x}{c} - \frac{1}{2} \sin \frac{2\pi x}{c} + \frac{1}{3} \sin \frac{3\pi x}{c} - \dots \right].$$

[-c < x < c.]

$$810. \quad x = \frac{c}{2} - \frac{4c}{\pi^2} \left[\cos \frac{\pi x}{c} + \frac{1}{3^2} \cos \frac{3\pi x}{c} + \frac{1}{5^2} \cos \frac{5\pi x}{c} + \dots \right].$$

[0 < x < c.]

$$\begin{aligned}
 811. \quad x^2 = \frac{2c^2}{\pi^3} & \left[\left(\frac{\pi^2}{1} - \frac{4}{1} \right) \sin \frac{\pi x}{c} - \frac{\pi^2}{2} \sin \frac{2\pi x}{c} \right. \\
 & + \left(\frac{\pi^2}{3} - \frac{4}{3^3} \right) \sin \frac{3\pi x}{c} - \frac{\pi^2}{4} \sin \frac{4\pi x}{c} \\
 & \left. + \left(\frac{\pi^2}{5} - \frac{4}{5^3} \right) \sin \frac{5\pi x}{c} + \dots \right].
 \end{aligned}$$

[0 < x < c.]

$$\begin{aligned}
 812. \quad x^2 = \frac{c^2}{3} - \frac{4c^2}{\pi^2} & \left[\cos \frac{\pi x}{c} - \frac{1}{2^2} \cos \frac{2\pi x}{c} + \frac{1}{3^2} \cos \frac{3\pi x}{c} \right. \\
 & \left. - \frac{1}{4^2} \cos \frac{4\pi x}{c} + \dots \right].
 \end{aligned}$$

[-c < x < c.]

$$813. \log \sin \frac{1}{2} x = -\log 2 - \cos x - \frac{1}{2} \cos 2x - \frac{1}{8} \cos 3x - \dots, \\ [0 < x < \frac{1}{2} \pi.]$$

$$814. \log \cos \frac{1}{2} x = -\log 2 + \cos x - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 3x - \dots, \\ [0 < x < \frac{1}{2} \pi.]$$

$$815. f(x) = \frac{1}{2} b_0 + b_1 \cos \frac{\pi x}{c} + b_2 \cos \frac{2\pi x}{c} + \dots \\ + a_1 \sin \frac{\pi x}{c} + a_2 \sin \frac{2\pi x}{c} + \dots, [-c < x < c.]$$

$$\text{where } b_m = \frac{1}{c} \int_{-c}^{+c} f(a) \cos \frac{m\pi a}{c} da,$$

$$a_m = \frac{1}{c} \int_{-c}^{+c} f(a) \sin \frac{m\pi a}{c} da.$$

$$816. \sin \theta = \theta \left[1 - \left(\frac{\theta}{\pi} \right)^2 \right] \left[1 - \left(\frac{\theta}{2\pi} \right)^2 \right] \left[1 - \left(\frac{\theta}{3\pi} \right)^2 \right] \dots, \\ [\theta^2 < \infty.]$$

$$817. \cos \theta = \left[1 - \left(\frac{2\theta}{\pi} \right)^2 \right] \left[1 - \left(\frac{2\theta}{3\pi} \right)^2 \right] \left[1 - \left(\frac{2\theta}{5\pi} \right)^2 \right] \dots, \\ [\theta^2 < \infty.]$$

$$818. \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2m)^2 (2m+2)}{1^2 \cdot 3^2 \cdot 5^2 \dots (2m+1)^2} > \frac{\pi}{2} \\ > \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2m)^2 (2m+1)}{1^2 \cdot 3^2 \cdot 5^2 \dots (2m+1)^2}.$$

$$819. J_n(x) = \frac{x^n}{2^n n!} \left\{ 1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4 (2n+2)(2n+4)} \right. \\ \left. - \frac{x^6}{2 \cdot 4 \cdot 6 (2n+2)(2n+4)(2n+6)} + \dots \right\}.$$

F. — DERIVATIVES.

$$820. \frac{d(au)}{dx} = a \frac{du}{dx}.$$

$$821. \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

$$822. \frac{d(uv)}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}.$$

$$823. \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

$$824. \frac{df(u)}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}.$$

$$825. \frac{d^2 f(u)}{dx^2} = \frac{df}{du} \cdot \frac{d^2 u}{dx^2} + \frac{d^2 f}{du^2} \cdot \frac{du^2}{dx^2}.$$

$$826. \frac{dx^n}{dx} = nx^{n-1}.$$

$$827. \frac{de^x}{dx} = e^x.$$

$$828. \frac{da^u}{dx} = a^u \cdot \frac{du}{dx} \cdot \log_e a.$$

$$829. \frac{dx^x}{dx} = x^x (1 + \log_e x).$$

$$830. \frac{d(\log_a x)}{dx} = \frac{1}{x \cdot \log_e a} = \frac{\log_a e}{x}$$

$$831. \frac{d \sin x}{dx} = \cos x.$$

$$832. \frac{d \cos x}{dx} = -\sin x.$$

$$833. \frac{d \tan x}{dx} = \sec^2 x.$$

$$834. \frac{d \operatorname{ctn} x}{dx} = -\operatorname{csc}^2 x.$$

$$835. \frac{d \sec x}{dx} = \tan x \cdot \sec x.$$

$$836. \frac{d \operatorname{csc} x}{dx} = -\operatorname{ctn} x \cdot \operatorname{csc} x.$$

$$837. \frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

$$838. \frac{d \cos^{-1} x}{dx} = \frac{-1}{\sqrt{1-x^2}}.$$

$$839. \frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}.$$

$$840. \frac{d \operatorname{ctn}^{-1} x}{dx} = -\frac{1}{1+x^2}.$$

$$841. \frac{d \sec^{-1} x}{dx} = \frac{1}{x\sqrt{x^2-1}}.$$

$$842. \frac{d \operatorname{csc}^{-1} x}{dx} = -\frac{1}{x\sqrt{x^2-1}}.$$

$$843. \frac{d \sinh x}{dx} = \cosh x.$$

$$844. \frac{d \cosh x}{dx} = \sinh x.$$

$$845. \frac{d \tanh x}{dx} = \operatorname{sech}^2 x.$$

$$846. \frac{d \operatorname{ctnh} x}{dx} = -\operatorname{csch}^2 x.$$

$$847. \frac{d \operatorname{sech} x}{dx} = -\operatorname{sech} x \cdot \tanh x.$$

$$848. \frac{d \operatorname{csch} x}{dx} = -\operatorname{csch} x \cdot \operatorname{ctnh} x.$$

$$849. \frac{d \sinh^{-1} x}{dx} = \frac{1}{\sqrt{x^2 + 1}}.$$

$$850. \frac{d \cosh^{-1} x}{dx} = \frac{1}{\sqrt{x^2 - 1}}.$$

$$851. \frac{d \tanh^{-1} x}{dx} = \frac{1}{1 - x^2}.$$

$$852. \frac{d \operatorname{ctnh}^{-1} x}{dx} = \frac{1}{1 - x^2}.$$

$$853. \frac{d \operatorname{sech}^{-1} x}{dx} = \frac{-1}{x \sqrt{1 - x^2}}.$$

$$854. \frac{d \operatorname{csch}^{-1} x}{dx} = \frac{-1}{x \sqrt{x^2 + 1}}.$$

$$855. \frac{d}{db} \int_a^b f(x) dx = f(b).$$

$$856. \frac{d}{da} \int_a^b f(x) dx = -f(a).$$

$$857. \frac{d}{dc} \int_a^b f(x, c) dx = \int_a^b D_c f(x, c) \cdot dx + f(b, c) \frac{db}{dc} - f(a, c) \frac{da}{dc}.$$

$$858. \frac{d^n (u \cdot v)}{dx^n} = v \cdot \frac{d^n u}{dx^n} + n \cdot \frac{dv}{dx} \cdot \frac{d^{n-1} u}{dx^{n-1}} \\ + \frac{n(n-1)}{2!} \cdot \frac{d^2 v}{dx^2} \cdot \frac{d^{n-2} u}{dx^{n-2}} + \cdots + u \frac{d^n v}{dx^n}.$$

859. If $f(x, y, z, \cdots)$ is a homogeneous function of the n th order, so that $f(\lambda x, \lambda y, \lambda z, \cdots) \equiv \lambda^n f(x, y, z, \cdots)$,

$$x \cdot D_x f + y \cdot D_y f + z \cdot D_z f + \cdots \equiv n f.$$

860. If $x = \phi(y)$,

$$\frac{dy}{dx} = \frac{1}{\phi'(y)}, \quad \frac{d^2y}{dx^2} = -\frac{\phi''(y)}{[\phi'(y)]^3},$$

$$\frac{d^3y}{dx^3} = \frac{3[\phi''(y)]^2 - \phi'(y) \cdot \phi'''(y)}{[\phi'(y)]^5}.$$

861. If $x = f(t)$ and $y = \phi(t)$,

$$\frac{dy}{dx} = \frac{\phi'(t)}{f'(t)}, \quad \frac{d^2y}{dx^2} = \frac{f'(t) \cdot \phi''(t) - f''(t) \cdot \phi'(t)}{[f'(t)]^3}.$$

862. If $f(x, y) = 0$,

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} / \frac{\partial f}{\partial y} \equiv -\frac{D_x f}{D_y f},$$

$$\frac{d^2y}{dx^2} = -\frac{D_x^2 f \cdot (D_y f)^2 - 2 D_x D_y f \cdot D_x f \cdot D_y f + D_y^2 f \cdot (D_x f)^2}{(D_y f)^3}.$$

863. If $y = f(u, v)$, $u = \phi(x)$, and $v = \psi(x)$,

$$\frac{df}{dx} = \frac{\partial f}{\partial u} \cdot \frac{du}{dx} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dx} = u' \cdot D_u f + v' \cdot D_v f,$$

$$\frac{d^2f}{dx^2} = \frac{\partial^2 f}{\partial u^2} \cdot \left(\frac{du}{dx}\right)^2 + 2 \frac{\partial^2 f}{\partial u \partial v} \cdot \frac{du}{dx} \cdot \frac{dv}{dx} + \frac{\partial^2 f}{\partial v^2} \cdot \left(\frac{dv}{dx}\right)^2$$

$$+ \frac{\partial f}{\partial u} \cdot \frac{d^2u}{dx^2} + \frac{\partial f}{\partial v} \cdot \frac{d^2v}{dx^2}$$

$$= u'^2 \cdot D_u^2 f + 2 u' \cdot v' \cdot D_u D_v f + v'^2 \cdot D_v^2 f$$

$$+ u'' \cdot D_u f + v'' \cdot D_v f.$$

864. If $f(x, y, z) = 0$, $D_x z = -D_x f / D_z f$,

$$D_x^2 z = -[D_x^2 f \cdot (D_z f)^2$$

$$- 2 D_z f \cdot D_x f \cdot D_x D_y f + D_z^2 f (D_x f)^2] / (D_z f)^3,$$

$$D_x D_y z = -[D_x D_y f \cdot (D_z f)^2 - D_z f D_x f \cdot D_y D_z f$$

$$+ D_z f \cdot D_y f \cdot D_x D_z f + D_x f \cdot D_y f \cdot D_z^2 f] / (D_z f)^3.$$

865. If $V = \phi(u, v)$, $u = f_1(x, y)$, and $v = f_2(x, y)$,

$$D_x V = D_u \phi \cdot D_x u + D_v \phi \cdot D_x v,$$

$$D_x^2 V = D_u^2 \phi \cdot (D_x u)^2 + D_v^2 \phi \cdot (D_x v)^2 + 2 D_u D_v \phi \cdot D_x u \cdot D_x v \\ + D_u \phi D_x^2 u + D_v \phi \cdot D_x^2 v,$$

$$D_y D_x V = D_u^2 \phi \cdot D_x u \cdot D_y u + D_v^2 \phi \cdot D_x v \cdot D_y v \\ + D_u D_v \phi (D_x v \cdot D_y u + D_x u \cdot D_y v) \\ + D_u \phi \cdot D_x D_y u + D_v \phi \cdot D_x D_y v,$$

$$D_x^2 V + D_y^2 V = D_u^2 \phi \cdot [(D_x u)^2 + (D_y u)^2] \\ + D_v^2 \phi \cdot [(D_x v)^2 + (D_y v)^2] \\ + 2 D_u D_v \phi \cdot [D_x u \cdot D_x v + D_y u \cdot D_y v] \\ + D_u \phi \cdot [D_x^2 u + D_y^2 u] \\ + D_v \phi \cdot [D_x^2 v + D_y^2 v].$$

In the special case, $u \equiv r \equiv \sqrt{x^2 + y^2}$, $v \equiv \theta \equiv \tan^{-1}(y/x)$, we have $D_x r = \cos \theta = x / \sqrt{x^2 + y^2}$; $D_y r = \sin \theta = y / \sqrt{x^2 + y^2}$;

$$D_\theta r = -r \sin \theta = -y; \quad D_\theta y = r \cos \theta = x;$$

$$D_x r = x / \sqrt{x^2 + y^2} = \cos \theta; \quad D_y r = y / \sqrt{x^2 + y^2} = \sin \theta;$$

$$D_x \theta = -y / (x^2 + y^2) = -\sin \theta / r;$$

$$D_y \theta = x / (x^2 + y^2) = \cos \theta / r; \quad \text{and}$$

$$D_x^2 V + D_y^2 V = D_r^2 V + \frac{1}{r} \cdot D_r V + \frac{1}{r^2} \cdot D_\theta^2 V.$$

866. If $V = \phi(u, v)$, $u = f_1(r, \theta)$, and $v = f_2(r, \theta)$,

$$D_r^2 V + \frac{1}{r} \cdot D_r V + \frac{1}{r^2} \cdot D_\theta^2 V = D_u^2 V \cdot \left[(D_r u)^2 + \frac{(D_\theta u)^2}{r^2} \right] \\ + D_v^2 V \cdot \left[(D_r v)^2 + \frac{(D_\theta v)^2}{r^2} \right] \\ + 2 D_u D_v V \left[D_r u \cdot D_r v + \frac{D_\theta u \cdot D_\theta v}{r^2} \right] +$$

$$\begin{aligned}
& + D_u V \left[D_r^2 u + \frac{1}{r} \cdot D_r u + \frac{1}{r^2} \cdot D_\theta^2 u \right] \\
& + D_v V \left[D_r^2 v + \frac{1}{r} \cdot D_r v + \frac{1}{r^2} \cdot D_\theta^2 v \right].
\end{aligned}$$

867. If $V = \phi(u, v, w)$, $u = f_1(x, y, z)$, $v = f_2(x, y, z)$, and
 $w = f_3(x, y, z)$,

$$D_x V = D_u V \cdot D_x u + D_v V \cdot D_x v + D_w V \cdot D_x w,$$

$$\begin{aligned}
D_x^2 V &= D_u^2 V \cdot (D_x u)^2 + D_v^2 V \cdot (D_x v)^2 + D_w^2 V \cdot (D_x w)^2 \\
&+ D_u V \cdot D_x^2 u + D_v V \cdot D_x^2 v + D_w V \cdot D_x^2 w \\
&+ 2(D_u D_v V \cdot D_x u \cdot D_x v + D_u D_w V \cdot D_x u \cdot D_x w \\
&+ D_v D_w V \cdot D_x v \cdot D_x w).
\end{aligned}$$

$$\begin{aligned}
D_x^2 V + D_y^2 V + D_z^2 V &= D_u^2 V \cdot [(D_x u)^2 + (D_y u)^2 + (D_z u)^2] \\
&+ D_v^2 V \cdot [(D_x v)^2 + (D_y v)^2 + (D_z v)^2] \\
&+ D_w^2 V \cdot [(D_x w)^2 + (D_y w)^2 + (D_z w)^2] \\
&+ 2 D_u D_v V \cdot [D_x u \cdot D_x v + D_y u \cdot D_y v + D_z u \cdot D_z v] \\
&+ 2 D_v D_w V \cdot [D_x v \cdot D_x w + D_y v \cdot D_y w + D_z v \cdot D_z w] \\
&+ 2 D_u D_w V \cdot [D_x u \cdot D_x w + D_y u \cdot D_y w + D_z u \cdot D_z w] \\
&+ D_u V \cdot [D_x^2 u + D_y^2 u + D_z^2 u] \\
&+ D_v V \cdot [D_x^2 v + D_y^2 v + D_z^2 v] \\
&+ D_w V \cdot [D_x^2 w + D_y^2 w + D_z^2 w].
\end{aligned}$$

In particular, if

$$x \equiv r \sin \theta \cos \phi, \quad y \equiv r \sin \theta \sin \phi, \quad z \equiv r \cos \theta,$$

so that $u \equiv r^2 \equiv x^2 + y^2 + z^2$, $v \equiv \theta \equiv \tan^{-1}(\sqrt{x^2 + y^2}/z)$,

$w \equiv \phi \equiv \tan^{-1}(y/x)$, we have

$$D_r z = \cos \theta = z / \sqrt{x^2 + y^2 + z^2};$$

$$D_r x = \sin \theta \cos \phi = x / \sqrt{x^2 + y^2 + z^2};$$

$$D_r y = \sin \theta \sin \phi = y / \sqrt{x^2 + y^2 + z^2};$$

$$D_\theta z = -r \sin \theta = -\sqrt{x^2 + y^2};$$

$$D_\theta x = r \cos \theta \cos \phi = zx / \sqrt{x^2 + y^2};$$

$$D_\theta y = r \cos \theta \sin \phi = zy / \sqrt{x^2 + y^2};$$

$$D_\phi z = 0;$$

$$D_\phi x = -r \sin \theta \sin \phi = -y;$$

$$D_\phi y = r \sin \theta \cos \phi = x;$$

$$D_z r = z / r = \cos \theta;$$

$$D_z \theta = -\sqrt{x^2 + y^2} / r^2 = -\sin \theta / r;$$

$$D_z \phi = 0;$$

$$D_x r = x / r = \sin \theta \cos \phi;$$

$$D_x \theta = xz / r^2 \sqrt{x^2 + y^2} = \cos \theta \cos \phi / r;$$

$$D_x \phi = -y / (x^2 + y^2) = -\sin \phi / r \sin \theta;$$

$$D_y r = y / r = \sin \theta \sin \phi;$$

$$D_y \theta = zy / r^2 \sqrt{x^2 + y^2} = \cos \theta \sin \phi / r;$$

$$D_y \phi = x / (x^2 + y^2) = \cos \phi / r \sin \theta;$$

$$(D_x r)^2 + (D_y r)^2 + (D_z r)^2 = 1;$$

$$(D_x \theta)^2 + (D_y \theta)^2 + (D_z \theta)^2 = 1 / r^2;$$

$$(D_x \phi)^2 + (D_y \phi)^2 + (D_z \phi)^2 = 1 / r^2 \sin^2 \theta;$$

$$\begin{aligned} (D_x V)^2 + (D_y V)^2 + (D_z V)^2 \\ = (D_r V)^2 + \left(\frac{D_\theta V}{r} \right)^2 + \left(\frac{D_\phi V}{r \sin \theta} \right)^2; \end{aligned}$$

$$D_x^2 V + D_y^2 V + D_z^2 V$$

$$= \frac{1}{r^2 \sin \theta} \left[D_r(r^2 \cdot D_r V) \cdot \sin \theta + \frac{D_\phi^2 V}{\sin \theta} + D_\theta(\sin \theta \cdot D_\theta V) \right].$$

868. If $x = f_1(u, v)$, $y = f_2(u, v)$, $z = f_3(u, v)$,

$$D_x z = \frac{D_u f_3 \cdot D_v f_2 - D_v f_3 \cdot D_u f_2}{D_u f_1 \cdot D_v f_2 - D_v f_1 \cdot D_u f_2},$$

$$D_y z = \frac{D_v f_3 \cdot D_u f_1 - D_u f_3 \cdot D_v f_1}{D_u f_1 \cdot D_v f_2 - D_v f_1 \cdot D_u f_2}.$$

869. If $x = f(z, u)$, and $y = \phi(z, u)$,

$$D_x z = D_u \phi / (D_z f \cdot D_u \phi - D_z \phi \cdot D_u f),$$

$$D_y z = D_u f / (D_z \phi \cdot D_u f - D_z f \cdot D_u \phi).$$

870. If $F_1(x, y, z, u, v) = 0$,

$$F_2(x, y, z, u, v) = 0, \text{ and } F_3(x, y, z, u, v) = 0,$$

$$D_x z \cdot \begin{vmatrix} D_z F_1 & D_u F_1 & D_v F_1 \\ D_z F_2 & D_u F_2 & D_v F_2 \\ D_z F_3 & D_u F_3 & D_v F_3 \end{vmatrix} = - \begin{vmatrix} D_x F_1 & D_u F_1 & D_v F_1 \\ D_x F_2 & D_u F_2 & D_v F_2 \\ D_x F_3 & D_u F_3 & D_v F_3 \end{vmatrix}.$$

871. If $F_1(x, y, z) = 0$, and $F_2(x, y, z) = 0$,

$$\frac{dy}{D_z F_1 \cdot D_x F_2 - D_z F_2 \cdot D_x F_1} = \frac{dz}{D_x F_1 \cdot D_y F_2 - D_x F_2 \cdot D_y F_1} = \frac{dx}{D_y F_1 \cdot D_z F_2 - D_y F_2 \cdot D_z F_1}.$$

If each of the quantities $y_1, y_2, y_3, \dots, y_n$ is a function of the n variables $x_1, x_2, x_3, \dots, x_n$, the determinant,

$$\begin{vmatrix} D_{x_1} y_1 & D_{x_2} y_1 & D_{x_3} y_1 & \cdots \\ D_{x_1} y_2 & D_{x_2} y_2 & D_{x_3} y_2 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ D_{x_1} y_n & D_{x_2} y_n & D_{x_3} y_n & \cdots & D_{x_n} y_n \end{vmatrix}$$

is called the *functional determinant* or the *Jacobian* of the y 's with respect to the x 's and is denoted by the expression,

$$\frac{\partial(y_1, y_2, y_3, \dots y_n)}{\partial(x_1, x_2, x_3, \dots x_n)}, \text{ or by } J(y_1, y_2, \dots y_n).$$

$$872. \frac{\partial(y_1, y_2, y_3, \dots y_n)}{\partial(x_1, x_2, x_3, \dots x_n)} \cdot \frac{\partial(x_1, x_2, x_3, \dots x_n)}{\partial(y_1, y_2, y_3, \dots y_n)} \equiv 1.$$

$$873. \frac{\partial(y_1, y_2, y_3, \dots y_n)}{\partial(z_1, z_2, z_3, \dots z_n)} \cdot \frac{\partial(z_1, z_2, z_3, \dots z_n)}{\partial(x_1, x_2, x_3, \dots x_n)} \\ \equiv \frac{\partial(y_1, y_2, y_3, \dots y_n)}{\partial(x_1, x_2, x_3, \dots x_n)}.$$

If the y 's are not all independent but are connected by an equation of the form $\phi(y_1, y_2, y_3, \dots y_n) = 0$, the Jacobian of the y 's with respect to the x 's vanishes identically; and, conversely, if the Jacobian vanishes identically, the y 's are connected by one or more relations of the above-mentioned form.

The *directional derivative* of any scalar point function, u , at any point, P , in any fixed direction PQ' , is the limit, as PQ approaches zero, of the ratio of $u_Q - u_P$ to PQ , where Q is a point on the straight line PQ' between P and Q' . The *gradient*, h_u , of the function u at P is the directional derivative of u at P taken in the direction in which u increases most rapidly. This direction is normal to the surface of constant u which passes through P .

$$874. h_u^2 \equiv (D_x u)^2 + (D_y u)^2 + (D_z u)^2.$$

The directional derivative of any scalar point function at any point in any given direction is evidently equal to the product of the gradient and the cosine of the angle between the given direction and that in which the function increases most rapidly.

The *normal derivative*, at any point, P , of a point function u , taken with respect to another point function v , is the limit as PQ approaches zero of the ratio of $u_Q - u_P$ to $v_Q - v_P$, where Q is a point so chosen on the normal at P of the surface of constant v which passes through P , that $v_Q - v_P$ is positive. If (u, v) denotes the angle between the directions in which u and v increase most rapidly, the normal derivatives of u with respect to v , and of v with respect to u may be written

$$h_u \cos (u, v) \div h_v, \text{ and } h_v \cdot \cos (u, v) \div h_u$$

respectively. If $h_u = h_v$, these derivatives are equal.

G. — MISCELLANEOUS FORMULAS.

If s is a plane analytic closed curve, n its normal drawn from within outwards, and dA the element of plane area within s , the usual integral transformation formulas for the functions u and v which, with their derivatives of the first order, are continuous everywhere within s , may be written —

$$875. \int u \cdot \cos (x, n) ds = \iint D_x u \cdot dA.$$

$$876. \int [u \cdot \cos (x, n) + v \cdot \cos (y, n)] ds = \iint (D_x u + D_y v) dA.$$

$$877. \int D_n u \cdot ds = \iint (D_x^2 u + D_y^2 u) dA.$$

$$\begin{aligned} 878. \iint (D_x u \cdot D_x v + D_y u \cdot D_y v) dA \\ = \int u \cdot D_n v \cdot ds - \iint u (D_x^2 v + D_y^2 v) dA \\ = \int v \cdot D_n u \cdot ds - \iint v (D_x^2 u + D_y^2 u) dA. \end{aligned}$$

$$\begin{aligned} 879. \iint \lambda (D_x u \cdot D_x v + D_y u \cdot D_y v) dA = \int \lambda \cdot u \cdot D_n v \cdot ds \\ - \iint u [D_x (\lambda \cdot D_x v) + D_y (\lambda \cdot D_y v)] dA \end{aligned}$$

If ξ and η are two analytic functions which define a set of orthogonal curvilinear coördinates, and if (ξ, n) and (η, n) represent the angles between n and the directions in which ξ and η , respectively, increase most rapidly.

$$880. \iint h_{\xi} \cdot h_{\eta} \cdot D_{\eta} \left(\frac{u}{h_{\xi}} \right) dA = \int u \cdot \cos (\eta, n) ds.$$

$$881. \iint h_{\xi} \cdot h_{\eta} \cdot D_{\xi} \left(\frac{u}{h_{\eta}} \right) dA = \int u \cdot \cos (\xi, n) ds.$$

882. If r is the distance from a fixed point, Q , in the coördinate plane,

$\int \frac{\cos (r, n) ds}{r} = 0, \pi$, or 2π , according as Q is without, on, or within s .

If S is an analytic closed surface, n its normal drawn from within outwards, and $d\tau$ the element of volume shut in by S , the usual integral transformation formulas may be written —

$$883. \iint u \cos (x, n) dS = \iiint D_x u \cdot d\tau.$$

$$884. \iint [u \cos (x, n) + v \cos (y, n) + w \cos (z, n)] dS \\ = \iiint (D_x u + D_y v + D_z w) d\tau.$$

$$885. \iint D_n u \cdot ds = \iiint (D_x^2 u + D_y^2 u + D_z^2 u) d\tau.$$

$$886. \iiint (D_x u \cdot D_x v + D_y u \cdot D_y v + D_z u \cdot D_z v) d\tau \\ = \iint u \cdot D_n v \cdot dS - \iiint u (D_x^2 v + D_y^2 v + D_z^2 v) d\tau \\ = \iint v \cdot D_n u \cdot dS - \iiint v (D_x^2 u + D_y^2 u + D_z^2 u) d\tau.$$

$$\begin{aligned}
 887. \quad & \iiint \lambda (D_x u \cdot D_x v + D_y u \cdot D_y v + D_z u \cdot D_z v) d\tau \\
 &= \iint \lambda \cdot v \cdot D_n u \cdot dS \\
 &- \iiint v [D_x (\lambda D_x u) + D_y (\lambda D_y u) + D_z (\lambda D_z u)] d\tau.
 \end{aligned}$$

If ξ, η, ζ are three analytic functions which define a system of orthogonal curvilinear coördinates,

$$888. \quad \iiint h_\xi \cdot h_\eta \cdot h_\zeta \cdot D_\xi \left(\frac{u}{h_\eta \cdot h_\zeta} \right) d\tau = \iint u \cdot \cos (\xi, n) dS.$$

$$889. \quad \iiint h_\xi \cdot h_\eta \cdot h_\zeta \cdot D_\eta \left(\frac{u}{h_\xi \cdot h_\zeta} \right) d\tau = \iint u \cdot \cos (\eta, n) dS.$$

$$890. \quad \iiint h_\xi \cdot h_\eta \cdot h_\zeta \cdot D_\zeta \left(\frac{u}{h_\xi \cdot h_\eta} \right) d\tau = \iint u \cdot \cos (\zeta, n) dS.$$

891. If r is the distance from a fixed point, Q ,

$\int \frac{\cos (r, n)}{r^2} dS = 0, 2\pi, \text{ or } 4\pi$ according as Q is without, on, or within S .

Stokes's Theorem. — The line integral, taken around a closed curve, of the tangential component of a vector point function, is equal to the surface integral, taken over a surface bounded by the curve, of the normal component of the curl of the vector, the direction of integration around the curve forming a right-handed screw rotation about the normals.

If X, Y, Z are the components of the vector,

$$\begin{aligned}
 892. \quad & \int (X dx + Y dy + Z dz) = \iint [(D_y Z - D_z Y) \cos (x, n) \\
 &+ (D_z X - D_x Z) \cos (y, n) \\
 &+ (D_x Y - D_y X) \cos (z, n)] dS.
 \end{aligned}$$

Equations 893 to 897 give Poisson's Equation in orthogonal Cartesian, in cylindrical, in spherical, and in orthogonal curvilinear coördinates.

$$893. \quad \bar{\nabla}^2 V \equiv D_x^2 V + D_y^2 V + D_z^2 V = -4\pi\rho.$$

$$894. \quad \frac{1}{r} \cdot D_r(r \cdot D_r V) + \frac{1}{r^2} \cdot D_\theta^2 V + D_z^2 V = -4\pi\rho.$$

$$895. \quad \sin \theta \cdot D_r(r^2 \cdot D_r V) + \frac{D_\phi^2 V}{\sin \theta} \\ + D_\theta(\sin \theta \cdot D_\theta V) = -4\pi\rho r^2 \sin \theta.$$

$$896. \quad h_\xi^2 \cdot D_\xi^2 V + h_\eta^2 \cdot D_\eta^2 V + h_\zeta^2 \cdot D_\zeta^2 V \\ + D_\xi V \cdot \bar{\nabla}^2 \xi + D_\eta V \cdot \bar{\nabla}^2 \eta + D_\zeta V \cdot \bar{\nabla}^2 \zeta = -4\pi\rho.$$

$$397. \quad h_\xi \cdot h_\eta \cdot h_\zeta \left\{ D_\xi \left(\frac{h_\xi}{h_\eta h_\zeta} \cdot D_\xi V \right) + D_\eta \left(\frac{h_\eta}{h_\xi h_\zeta} \cdot D_\eta V \right) \right. \\ \left. + D_\zeta \left(\frac{h_\zeta}{h_\xi h_\eta} \cdot D_\zeta V \right) \right\} = -4\pi\rho$$

H. — CERTAIN CONSTANTS.

$$\pi = 3.14159 \ 26535 \ 89793$$

$$\log_{10} \pi = 0.49714 \ 98726 \ 94134$$

$$\frac{1}{\pi} = 0.31830 \ 98861 \ 83791$$

$$\pi^2 = 9.86960 \ 44010 \ 89359$$

$$\sqrt{\pi} = 1.77245 \ 38509 \ 05516$$

$$\log_{10} 2 = 0.30102 \ 99956 \ 63981$$

$$e = 2.71828 \ 18284 \ 59045$$

$$\log_{10} e = 0.43429 \ 44819 \ 03252$$

$$\log_e 10 = 2.30258 \ 50929 \ 94046$$

$$\log_e 2 = 0.69314 \ 71805 \ 59945$$

$$\log_{10} \log_{10} e = 9.63778 \ 43113 \ 00537$$

$$\log_e \pi = 1.14472 \ 98858 \ 49400$$

I. — GENERAL FORMULAS OF INTEGRATION.

F and f represent functions of x , and F' , f' , F'' , f'' , their first and second derivatives with respect to x .

$$898. \int F' \cdot f \cdot dx = F \cdot f - \int F \cdot f' \cdot dx.$$

$$899. \int (F)^n \cdot F' \cdot dx = (F)^{n+1}/(n+1).$$

$$900. \int (aF + b)^n \cdot F' \cdot dx = (aF + b)^{n+1}/a(n+1).$$

$$901. \int (F + f)^n \cdot dx = \int F(F + f)^{n-1} dx + \int f(F + f)^{n-1} dx.$$

$$902. \int F'/(F)^n \cdot dx = -1/(n-1)(F)^{n-1}, \quad \int F'/F \cdot dx = \log F.$$

$$903. \int (F' \cdot f - F \cdot f')/(f)^2 \cdot dx = F/f.$$

$$904. \int (F' \cdot f - F \cdot f')/Ff \cdot dx = \log (F/f).$$

$$905. \int \frac{dx}{F \cdot (x^2 - a^2)} = \frac{1}{2a} \int \frac{dx}{F \cdot (x - a)} - \frac{1}{2a} \int \frac{dx}{F \cdot (x + a)}.$$

$$906. \int \frac{dx}{F(F \pm f)} = \pm \int \frac{dx}{F \cdot f} \mp \int \frac{dx}{f(F \pm f)}.$$

$$907. \int \frac{F' \cdot dx}{\sqrt{aF + b}} = (2\sqrt{aF + b})/a.$$

$$908. \int \frac{F' \cdot dx}{\sqrt{F^2 + a}} = \log (F + \sqrt{F^2 + a}).$$

$$909. \int \frac{F \cdot dx}{(F + a)(F + b)} = \frac{a}{a - b} \int \frac{dx}{F + a} - \frac{b}{a - b} \int \frac{dx}{F + b}.$$

$$910. \int \frac{F \cdot dx}{(F + f)^n} = \int \frac{dx}{(F + f)^{n-1}} - \int \frac{f dx}{(F + f)^n}.$$

$$911. \int \frac{F' \cdot dx}{p^2 + q^2 F^2} = \frac{1}{pq} \cdot \tan^{-1} \frac{qF}{p}, \quad \int \frac{F' \cdot dx}{q^2 F^2 - p^2} = \frac{1}{2pq} \log \frac{qF - p}{qF + p}.$$

$$912. \int \frac{F^{2n} \cdot dx}{1 - F^{2n}} = -x + \int \frac{dx}{1 - F^{2n}}.$$

$$913. \int \frac{F' \cdot dx}{F'^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{F}{a} \right).$$

$$914. \int \frac{F' \cdot dx}{a^2 F'^2 - b^2} = \frac{1}{2ab} \log \frac{aF - b}{aF + b}.$$

$$915. \int \frac{F^{2n} \cdot dx}{F^{2n} - b^2} = \int \frac{F^n \cdot dx}{2(F^n - b)} + \int \frac{F^n \cdot dx}{2(F^n + b)}.$$

$$916. \int \frac{F' \cdot dx}{\sqrt{b^2 - F'^2}} = \sin^{-1} \left(\frac{F}{b} \right).$$

$$917. \int \frac{F' \cdot dx}{aF'^2 + bF'} = \frac{1}{b} \log \frac{F}{aF + b}.$$

$$918. \int \frac{F' \cdot dx}{aF'^2 - bF'} = \frac{1}{b} \log \frac{aF - b}{F'}.$$

$$919. \int \frac{F'}{F' \sqrt{F'^2 - b^2}} = \frac{1}{b} \sec^{-1} \left(\frac{F}{b} \right).$$

$$920. \int \frac{(F' \cdot f - F \cdot f') dx}{F'^2 + f^2} = \tan^{-1} \left(\frac{F}{f} \right).$$

$$921. \int \frac{(F' \cdot f - F \cdot f') dx}{F'^2 - f^2} = \frac{1}{2} \log \left(\frac{F - f}{F + f} \right).$$

J — INTEGRALS USEFUL IN THE THEORY OF ALTERNATING CURRENTS.

$$922. \int \sin (\omega t + \phi) dt = -\frac{1}{\omega} \cdot \cos (\omega t + \phi).$$

$$923. \int \cos (\omega t + \phi) dt = \frac{1}{\omega} \cdot \sin (\omega t + \phi).$$

$$924. \int \sin^2 (\omega t + \phi) dt = \frac{1}{2} t - \frac{1}{4\omega} \sin 2(\omega t + \phi).$$

$$925. \int \sin (\omega t + \phi) \cdot \cos (\omega t + \phi) dt = \frac{1}{2 \omega} \cdot \sin ^2 (\omega t + \phi).$$

$$926. \int \cos ^2 (\omega t + \phi) dt = \frac{1}{2} t + \frac{1}{4 \omega} \sin 2 (\omega t + \phi).$$

$$927. \int \sin (\omega t + \lambda) \cdot \sin (\omega t + \mu) dt = \frac{\cos (\mu - \lambda)}{2 \omega} (\omega t) \\ - \frac{\sin (\omega t + \lambda) \cdot \cos (\omega t + \mu)}{2 \omega}$$

$$928. \int \sin (\omega t + \lambda) \cdot \cos (\omega t + \mu) dt = \frac{\sin (\omega t + \lambda) \cdot \sin (\omega t + \mu)}{2 \omega} \\ - \frac{\sin (\mu - \lambda)}{2 \omega} (\omega t).$$

$$929. \int \cos (\omega t + \lambda) \cdot \cos (\omega t + \mu) dt = \frac{\cos (\mu - \lambda)}{2 \omega} (\omega t) \\ + \frac{\sin (\omega t + \lambda) \cdot \cos (\omega t + \lambda)}{2 \omega}$$

$$930. \int \sin (mt + \lambda) \cdot \sin (nt + \mu) dt = \frac{\sin [mt - nt + \lambda - \mu]}{2(m - n)} \\ - \frac{\sin [mt + nt + \lambda + \mu]}{2(m + n)}.$$

$$931. \int \cos (mt + \lambda) \cdot \cos (nt + \mu) dt = \frac{\sin [mt + nt + \lambda + \mu]}{2(m + n)} \\ + \frac{\sin [mt - nt + \lambda - \mu]}{2(m - n)}.$$

$$932. \int \sin (mt + \lambda) \cdot \cos (nt + \mu) dt = - \frac{\cos [mt + nt + \lambda + \mu]}{2(m + n)} \\ - \frac{\cos [mt - nt + \lambda - \mu]}{2(m - n)}.$$

$$\begin{aligned}
 933. \quad & \int \cos(\omega t + \lambda + mx) \cdot \cos(\omega t + \lambda - mx) dx \\
 &= \cos^2(\omega t + \lambda) \left[\frac{mx + \sin mx \cdot \cos mx}{2m} \right] \\
 &\quad - \sin^2(\omega t + \lambda) \left[\frac{mx - \sin mx \cdot \cos mx}{2m} \right]. \\
 &\left\{ \begin{aligned} m \cdot \sin(\omega t + \phi) + n \cdot \cos(\omega t + \phi) &= \sqrt{m^2 + n^2} \cdot \sin(\omega t + \phi + \hat{c}) \\ \text{where } \tan \hat{c} &= n/m. \\ m \cdot \sin(\omega t + \phi) - n \cdot \cos(\omega t + \phi) &= \sqrt{m^2 + n^2} \cdot \sin(\omega t + \phi - \hat{c}). \end{aligned} \right\}
 \end{aligned}$$

$$\begin{aligned}
 934. \quad & \int e^{(-b \pm ci)t} dt = \frac{-b \mp ci}{b^2 + c^2} e^{(-b \pm ci)t} \\
 &= \frac{e^{-bt}}{b^2 + c^2} [(c \cdot \sin ct - b \cdot \cos ct) \mp i (b \cdot \sin ct + c \cdot \cos ct)] \\
 &= \frac{e^{-bt}}{\sqrt{b^2 + c^2}} [\sin(ct - \delta) \mp i \cdot \cos(ct - \delta)], \\
 &\hspace{15em} \text{where } \tan \delta = b/c.
 \end{aligned}$$

$$\begin{aligned}
 935. \quad & \int e^{\alpha t} \cdot \cos(\omega t + \phi) dt \\
 &= \frac{e^{\alpha t}}{\alpha^2 + \omega^2} [\omega \sin(\omega t + \phi) + \alpha \cdot \cos(\omega t + \phi)] \\
 &= \frac{e^{\alpha t}}{\sqrt{\alpha^2 + \omega^2}} \cos[\omega t + \phi - \tan^{-1}(\omega/\alpha)].
 \end{aligned}$$

$$\begin{aligned}
 936. \quad & \int e^{\alpha t} \cdot \sin(\omega t + \phi) dt \\
 &= \frac{e^{\alpha t}}{\alpha^2 + \omega^2} [\alpha \cdot \sin(\omega t + \phi) - \omega \cdot \cos(\omega t + \phi)] \\
 &= \frac{e^{\alpha t}}{\sqrt{\alpha^2 + \omega^2}} \sin[\omega t + \phi - \tan^{-1}(\omega/\alpha)].
 \end{aligned}$$

$$\begin{aligned}
 937. \quad & \int [e^{\alpha t} \cdot \sin(\omega t + \phi)]^2 dt \\
 &= \frac{e^{2\alpha t}}{4} \left[\frac{1}{\alpha} - \frac{\omega \cdot \sin 2(\omega t + \phi) + \alpha \cdot \cos 2(\omega t + \phi)}{\alpha^2 + \omega^2} \right] \\
 &= \frac{e^{2\alpha t}}{4} \left[\frac{1}{\alpha} - \frac{\cos[2\omega t + 2\phi - \tan^{-1}(\omega/\alpha)]}{\sqrt{\alpha^2 + \omega^2}} \right].
 \end{aligned}$$

$$\begin{aligned}
 938. \quad & \int [e^{at} \cdot \cos(\omega t + \phi)]^2 dt \\
 &= \frac{e^{2at}}{4} \left[\frac{1}{a} + \frac{\omega \cdot \sin 2(\omega t + \phi) + \alpha \cdot \cos 2(\omega t + \phi)}{a^2 + \omega^2} \right] \\
 &= \frac{e^{2at}}{4} \left[\frac{1}{a} + \frac{\cos [2\omega t + 2\phi - \tan^{-1}(\omega/\alpha)]}{\sqrt{a^2 + \omega^2}} \right].
 \end{aligned}$$

In the case of a direct trigonometric function of $(\omega t + \phi)$, $T = 2\pi/\omega$ is called the *period* or the *cycle*. The mean value for any whole number of periods, reckoned from any epoch, of $\sin(\omega t + \phi)$, $\cos(\omega t + \phi)$, or $\sin(\omega t + \phi) \cdot \cos(\omega t + \phi)$, is zero, whereas the mean value for any whole number of half periods, reckoned from any epoch, of either $\sin^2(\omega t + \phi)$ or $\cos^2(\omega t + \phi)$ is one half. The mean value of $\sin(\omega t)$ from $t = 0$ to $t = \frac{1}{2} T$, or of $\cos(\omega t)$ from $-\frac{1}{4} T$ to $+\frac{1}{4} T$, is $2/\pi$ or 0.6366.

The mean value, for any number of whole periods, of either $\sin(\omega t + \lambda) \cdot \sin(\omega t + \mu)$ or $\cos(\omega t + \lambda) \cdot \cos(\omega t + \mu)$ is $\frac{1}{2} \cdot \cos(\lambda - \mu)$, while the mean value of $\sin(\omega t + \lambda) \cdot \cos(\omega t + \mu)$ is $\frac{1}{2} \sin(\lambda - \mu)$.

INTERPOLATION.

If values of an analytic function, $f(x)$, are given in a table for a number of values of the argument x , separated from one another consecutively by the constant small interval, δ , the differences between successive tabular values of the function are called *first tabular differences*, the differences of these first differences, *second tabular differences*, and so on. The tabular differences of the first, second, third, and fourth orders corresponding to $x = a$ are

$$\Delta_1 \equiv f(a + \delta) - f(a),$$

$$\Delta_2 \equiv f(a + 2\delta) - 2 \cdot f(a + \delta) + f(a),$$

$$\Delta_3 \equiv f(a + 3\delta) - 3 \cdot f(a + 2\delta) + 3 \cdot f(a + \delta) - f(a),$$

$$\Delta_4 \equiv f(a + 4\delta) - 4 \cdot f(a + 3\delta) + 6 \cdot f(a + 2\delta) - 4 \cdot f(a + \delta) + f(a),$$

where $f(a)$ is any tabulated value.

The value of the function for $x = (a + h)$, where $h = k\delta$, is

$$\begin{aligned} f(a + h) = f(a) + k \cdot \Delta_1 + \frac{k(k-1)}{2!} \cdot \Delta_2 + \frac{k(k-1)(k-2)}{3!} \cdot \Delta_3 \\ + \frac{k(k-1)(k-2)(k-3)}{4!} \cdot \Delta_4 + \dots \end{aligned}$$

The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx. \right)$$

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.00 | 0.00000 | 00113 | 00226 | 00339 | 00451 | 00564 | 00677 | 00790 | 00903 | 01016 |
| 0.01 | 0.01128 | 01241 | 01354 | 01467 | 01580 | 01792 | 01805 | 01918 | 02031 | 02144 |
| 0.02 | 0.02256 | 02369 | 02482 | 02595 | 02708 | 02820 | 02933 | 03046 | 03159 | 03271 |
| 0.03 | 0.03384 | 03497 | 03610 | 03722 | 03835 | 03948 | 04060 | 04173 | 04286 | 04398 |
| 0.04 | 0.04511 | 04624 | 04736 | 04849 | 04962 | 05074 | 05187 | 05299 | 05412 | 05525 |
| 0.05 | 0.05637 | 05750 | 05862 | 05975 | 06087 | 06200 | 06312 | 06425 | 06537 | 06650 |
| 0.06 | 0.06762 | 06875 | 06987 | 07099 | 07212 | 07324 | 07437 | 07549 | 07661 | 07773 |
| 0.07 | 0.07886 | 07998 | 08110 | 08223 | 08335 | 08447 | 08559 | 08671 | 08784 | 08896 |
| 0.08 | 0.09008 | 09120 | 09232 | 09344 | 09456 | 09568 | 09680 | 09792 | 09904 | 10016 |
| 0.09 | 0.10128 | 10240 | 10352 | 10464 | 10576 | 10687 | 10799 | 10911 | 11023 | 11135 |
| 0.10 | 0.11246 | 11358 | 11470 | 11581 | 11693 | 11805 | 11916 | 12028 | 12139 | 12251 |
| 0.11 | 0.12362 | 12474 | 12585 | 12697 | 12808 | 12919 | 13031 | 13142 | 13253 | 13365 |
| 0.12 | 0.13476 | 13587 | 13698 | 13809 | 13921 | 14032 | 14143 | 14254 | 14365 | 14476 |
| 0.13 | 0.14587 | 14698 | 14809 | 14919 | 15030 | 15141 | 15252 | 15363 | 15473 | 15584 |
| 0.14 | 0.15695 | 15805 | 15916 | 16027 | 16137 | 16248 | 16358 | 16468 | 16579 | 16689 |
| 0.15 | 0.16800 | 16910 | 17020 | 17130 | 17241 | 17351 | 17461 | 17571 | 17681 | 17791 |
| 0.16 | 0.17901 | 18011 | 18121 | 18231 | 18341 | 18451 | 18560 | 18670 | 18780 | 18890 |
| 0.17 | 0.18999 | 19109 | 19218 | 19328 | 19437 | 19547 | 19656 | 19766 | 19875 | 19984 |
| 0.18 | 0.20094 | 20203 | 20312 | 20421 | 20530 | 20639 | 20748 | 20857 | 20966 | 21075 |
| 0.19 | 0.21184 | 21293 | 21402 | 21510 | 21619 | 21728 | 21836 | 21945 | 22053 | 22162 |
| 0.20 | 0.22270 | 22379 | 22487 | 22595 | 22704 | 22812 | 22920 | 23028 | 23136 | 23244 |
| 0.21 | 0.23352 | 23460 | 23568 | 23676 | 23784 | 23891 | 23999 | 24107 | 24214 | 24322 |
| 0.22 | 0.24430 | 24537 | 24645 | 24752 | 24859 | 24967 | 25074 | 25181 | 25288 | 25395 |
| 0.23 | 0.25502 | 25609 | 25716 | 25823 | 25930 | 26037 | 26144 | 26250 | 26357 | 26463 |
| 0.24 | 0.26570 | 26677 | 26783 | 26889 | 26996 | 27102 | 27208 | 27314 | 27421 | 27527 |
| 0.25 | 0.27633 | 27739 | 27845 | 27950 | 28056 | 28162 | 28268 | 28373 | 28479 | 28584 |
| 0.26 | 0.28690 | 28795 | 28901 | 29006 | 29111 | 29217 | 29322 | 29427 | 29532 | 29637 |
| 0.27 | 0.29742 | 29847 | 29952 | 30056 | 30161 | 30266 | 30370 | 30475 | 30579 | 30684 |
| 0.28 | 0.30788 | 30892 | 30997 | 31101 | 31205 | 31309 | 31413 | 31517 | 31621 | 31725 |
| 0.29 | 0.31828 | 31922 | 32036 | 32139 | 32243 | 32346 | 32450 | 32553 | 32656 | 32760 |
| 0.30 | 0.32863 | 32966 | 33069 | 33172 | 33275 | 33378 | 33480 | 33583 | 33686 | 33788 |
| 0.31 | 0.33891 | 33993 | 34096 | 34198 | 34300 | 34403 | 34505 | 34607 | 34709 | 34811 |
| 0.32 | 0.34913 | 35014 | 35116 | 35218 | 35319 | 35421 | 35523 | 35624 | 35725 | 35827 |
| 0.33 | 0.35928 | 36029 | 36130 | 36231 | 36332 | 36433 | 36534 | 36635 | 36735 | 36836 |
| 0.34 | 0.36936 | 37037 | 37137 | 37238 | 37338 | 37438 | 37538 | 37638 | 37738 | 37838 |
| 0.35 | 0.37938 | 38038 | 38138 | 38237 | 38337 | 38436 | 38536 | 38635 | 38735 | 38834 |
| 0.36 | 0.38933 | 39032 | 39131 | 39230 | 39329 | 39428 | 39526 | 39625 | 39724 | 39822 |
| 0.37 | 0.39921 | 40019 | 40117 | 40215 | 40314 | 40412 | 40510 | 40608 | 40705 | 40803 |
| 0.38 | 0.40901 | 40999 | 41096 | 41194 | 41291 | 41388 | 41486 | 41583 | 41680 | 41777 |
| 0.39 | 0.41874 | 41971 | 42068 | 42164 | 42261 | 42358 | 42454 | 42550 | 42647 | 42743 |
| 0.40 | 0.42839 | 42935 | 43031 | 43127 | 43223 | 43319 | 43415 | 43510 | 43606 | 43701 |
| 0.41 | 0.43797 | 43892 | 43988 | 44083 | 44178 | 44273 | 44368 | 44463 | 44557 | 44652 |
| 0.42 | 0.44747 | 44841 | 44936 | 45030 | 45124 | 45219 | 45313 | 45407 | 45501 | 45595 |
| 0.43 | 0.45689 | 45782 | 45876 | 45970 | 46063 | 46157 | 46250 | 46343 | 46436 | 46529 |
| 0.44 | 0.46623 | 46715 | 46808 | 46901 | 46994 | 47086 | 47179 | 47271 | 47364 | 47456 |
| 0.45 | 0.47548 | 47640 | 47732 | 47824 | 47916 | 48008 | 48100 | 48191 | 48283 | 48374 |
| 0.46 | 0.48466 | 48557 | 48648 | 48739 | 48830 | 48921 | 49012 | 49103 | 49193 | 49284 |
| 0.47 | 0.49375 | 49465 | 49555 | 49646 | 49736 | 49826 | 49916 | 50006 | 50096 | 50185 |
| 0.48 | 0.50275 | 50365 | 50454 | 50543 | 50633 | 50722 | 50811 | 50900 | 50989 | 51078 |
| 0.49 | 0.51167 | 51256 | 51344 | 51433 | 51521 | 51609 | 51698 | 51786 | 51874 | 51962 |

The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx. \right)$$

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.50 | 0.52050 | 52138 | 52226 | 52313 | 52401 | 52488 | 52576 | 52663 | 52750 | 52837 |
| 0.51 | 0.52924 | 53011 | 53098 | 53185 | 53272 | 53358 | 53445 | 53531 | 53617 | 53704 |
| 0.52 | 0.53790 | 53876 | 53962 | 54048 | 54134 | 54219 | 54305 | 54390 | 54476 | 54561 |
| 0.53 | 0.54646 | 54732 | 54817 | 54902 | 54987 | 55071 | 55156 | 55241 | 55325 | 55410 |
| 0.54 | 0.55494 | 55578 | 55662 | 55746 | 55830 | 55914 | 55998 | 56082 | 56165 | 56249 |
| 0.55 | 0.56332 | 56416 | 56499 | 56582 | 56665 | 56748 | 56831 | 56914 | 56996 | 57079 |
| 0.56 | 0.57162 | 57244 | 57326 | 57409 | 57491 | 57573 | 57655 | 57737 | 57818 | 57900 |
| 0.57 | 0.57982 | 58063 | 58144 | 58226 | 58307 | 58388 | 58469 | 58550 | 58631 | 58712 |
| 0.58 | 0.58792 | 58873 | 58953 | 59034 | 59114 | 59194 | 59274 | 59354 | 59434 | 59514 |
| 0.59 | 0.59594 | 59673 | 59753 | 59832 | 59912 | 59991 | 60070 | 60149 | 60228 | 60307 |
| 0.60 | 0.60386 | 60464 | 60543 | 60621 | 60700 | 60778 | 60856 | 60934 | 61012 | 61090 |
| 0.61 | 0.61168 | 61246 | 61323 | 61401 | 61478 | 61556 | 61633 | 61710 | 61787 | 61864 |
| 0.62 | 0.61941 | 62018 | 62095 | 62171 | 62248 | 62324 | 62400 | 62477 | 62553 | 62629 |
| 0.63 | 0.62705 | 62780 | 62856 | 62932 | 63007 | 63083 | 63158 | 63233 | 63309 | 63384 |
| 0.64 | 0.63459 | 63533 | 63608 | 63683 | 63757 | 63832 | 63906 | 63981 | 64055 | 64129 |
| 0.65 | 0.64203 | 64277 | 64351 | 64424 | 64498 | 64572 | 64645 | 64718 | 64791 | 64865 |
| 0.66 | 0.64938 | 65011 | 65083 | 65156 | 65229 | 65301 | 65374 | 65446 | 65519 | 65591 |
| 0.67 | 0.65663 | 65735 | 65807 | 65878 | 65950 | 66022 | 66093 | 66165 | 66236 | 66307 |
| 0.68 | 0.66378 | 66449 | 66520 | 66591 | 66662 | 66732 | 66803 | 66873 | 66944 | 67014 |
| 0.69 | 0.67084 | 67154 | 67224 | 67294 | 67364 | 67433 | 67503 | 67572 | 67642 | 67711 |
| 0.70 | 0.67780 | 67849 | 67918 | 67987 | 68056 | 68125 | 68193 | 68262 | 68330 | 68398 |
| 0.71 | 0.68467 | 68535 | 68603 | 68671 | 68738 | 68806 | 68874 | 68941 | 69009 | 69076 |
| 0.72 | 0.69143 | 69210 | 69278 | 69344 | 69411 | 69478 | 69545 | 69611 | 69678 | 69744 |
| 0.73 | 0.69810 | 69877 | 69943 | 70009 | 70075 | 70140 | 70206 | 70272 | 70337 | 70403 |
| 0.74 | 0.70468 | 70533 | 70598 | 70663 | 70728 | 70793 | 70858 | 70922 | 70987 | 71051 |
| 0.75 | 0.71116 | 71180 | 71244 | 71308 | 71372 | 71436 | 71500 | 71563 | 71627 | 71690 |
| 0.76 | 0.71754 | 71817 | 71880 | 71943 | 72006 | 72069 | 72132 | 72195 | 72257 | 72320 |
| 0.77 | 0.72382 | 72444 | 72507 | 72569 | 72631 | 72693 | 72755 | 72816 | 72878 | 72940 |
| 0.78 | 0.73001 | 73062 | 73124 | 73185 | 73246 | 73307 | 73368 | 73429 | 73489 | 73550 |
| 0.79 | 0.73610 | 73671 | 73731 | 73791 | 73851 | 73911 | 73971 | 74031 | 74091 | 74151 |
| 0.80 | 0.74210 | 74270 | 74329 | 74388 | 74447 | 74506 | 74565 | 74624 | 74683 | 74742 |
| 0.81 | 0.74800 | 74859 | 74917 | 74976 | 75034 | 75092 | 75150 | 75208 | 75266 | 75323 |
| 0.82 | 0.75381 | 75439 | 75496 | 75553 | 75611 | 75668 | 75725 | 75782 | 75839 | 75896 |
| 0.83 | 0.75952 | 76009 | 76066 | 76122 | 76178 | 76234 | 76291 | 76347 | 76403 | 76459 |
| 0.84 | 0.76514 | 76570 | 76626 | 76681 | 76736 | 76792 | 76847 | 76902 | 76957 | 77012 |
| 0.85 | 0.77067 | 77122 | 77176 | 77231 | 77285 | 77340 | 77394 | 77448 | 77502 | 77556 |
| 0.86 | 0.77610 | 77664 | 77718 | 77771 | 77825 | 77878 | 77932 | 77985 | 78038 | 78091 |
| 0.87 | 0.78144 | 78197 | 78250 | 78302 | 78355 | 78408 | 78460 | 78512 | 78565 | 78617 |
| 0.88 | 0.78669 | 78721 | 78773 | 78824 | 78876 | 78928 | 78979 | 79031 | 79082 | 79133 |
| 0.89 | 0.79184 | 79235 | 79286 | 79337 | 79388 | 79439 | 79489 | 79540 | 79590 | 79641 |
| 0.90 | 0.79691 | 79741 | 79791 | 79841 | 79891 | 79941 | 79990 | 80040 | 80090 | 80139 |
| 0.91 | 0.80188 | 80238 | 80287 | 80336 | 80385 | 80434 | 80482 | 80531 | 80580 | 80628 |
| 0.92 | 0.80677 | 80725 | 80773 | 80822 | 80870 | 80918 | 80966 | 81013 | 81061 | 81109 |
| 0.93 | 0.81156 | 81204 | 81251 | 81299 | 81346 | 81393 | 81440 | 81487 | 81534 | 81580 |
| 0.94 | 0.81627 | 81674 | 81720 | 81767 | 81813 | 81859 | 81905 | 81951 | 81997 | 82043 |
| 0.95 | 0.82089 | 82135 | 82180 | 82226 | 82271 | 82317 | 82362 | 82407 | 82452 | 82497 |
| 0.96 | 0.82542 | 82587 | 82632 | 82677 | 82721 | 82766 | 82810 | 82855 | 82899 | 82943 |
| 0.97 | 0.82987 | 83031 | 83075 | 83119 | 83162 | 83206 | 83250 | 83293 | 83337 | 83380 |
| 0.98 | 0.83423 | 83466 | 83509 | 83552 | 83595 | 83638 | 83681 | 83723 | 83766 | 83808 |
| 0.99 | 0.83851 | 83893 | 83935 | 83977 | 84020 | 84061 | 84103 | 84145 | 84187 | 84229 |

The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx. \right)$$

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1.00 | 0.84270 | 84312 | 84353 | 84394 | 84435 | 84477 | 84518 | 84559 | 84600 | 84640 |
| 1.01 | 0.84681 | 84722 | 84762 | 84803 | 84843 | 84883 | 84924 | 84964 | 85004 | 85044 |
| 1.02 | 0.85084 | 85124 | 85163 | 85203 | 85243 | 85282 | 85322 | 85361 | 85400 | 85439 |
| 1.03 | 0.85478 | 85517 | 85556 | 85595 | 85634 | 85673 | 85711 | 85750 | 85788 | 85827 |
| 1.04 | 0.85865 | 85903 | 85941 | 85979 | 86017 | 86055 | 86093 | 86131 | 86169 | 86206 |
| 1.05 | 0.86244 | 86281 | 86318 | 86356 | 86393 | 86430 | 86467 | 86504 | 86541 | 86578 |
| 1.06 | 0.86614 | 86651 | 86688 | 86724 | 86760 | 86797 | 86833 | 86869 | 86905 | 86941 |
| 1.07 | 0.86977 | 87013 | 87049 | 87085 | 87120 | 87156 | 87191 | 87227 | 87262 | 87297 |
| 1.08 | 0.87333 | 87368 | 87403 | 87438 | 87473 | 87507 | 87542 | 87577 | 87611 | 87646 |
| 1.09 | 0.87680 | 87715 | 87749 | 87783 | 87817 | 87851 | 87885 | 87919 | 87953 | 87987 |
| 1.10 | 0.88021 | 88054 | 88088 | 88121 | 88155 | 88188 | 88221 | 88254 | 88287 | 88320 |
| 1.11 | 0.88353 | 88386 | 88419 | 88452 | 88484 | 88517 | 88549 | 88582 | 88614 | 88647 |
| 1.12 | 0.88679 | 88711 | 88743 | 88775 | 88807 | 88839 | 88871 | 88902 | 88934 | 88966 |
| 1.13 | 0.88997 | 89029 | 89060 | 89091 | 89122 | 89154 | 89185 | 89216 | 89247 | 89277 |
| 1.14 | 0.89308 | 89339 | 89370 | 89400 | 89431 | 89461 | 89492 | 89522 | 89552 | 89582 |
| 1.15 | 0.89612 | 89642 | 89672 | 89702 | 89732 | 89762 | 89792 | 89821 | 89851 | 89880 |
| 1.16 | 0.89910 | 89939 | 89968 | 89997 | 90027 | 90056 | 90085 | 90114 | 90142 | 90171 |
| 1.17 | 0.90200 | 90229 | 90257 | 90286 | 90314 | 90343 | 90371 | 90399 | 90428 | 90456 |
| 1.18 | 0.90484 | 90512 | 90540 | 90568 | 90595 | 90623 | 90651 | 90678 | 90706 | 90733 |
| 1.19 | 0.90761 | 90788 | 90815 | 90843 | 90870 | 90897 | 90924 | 90951 | 90978 | 91005 |
| 1.20 | 0.91031 | 91058 | 91085 | 91111 | 91138 | 91164 | 91191 | 91217 | 91243 | 91269 |
| 1.21 | 0.91296 | 91322 | 91348 | 91374 | 91399 | 91425 | 91451 | 91477 | 91502 | 91528 |
| 1.22 | 0.91553 | 91579 | 91604 | 91630 | 91655 | 91680 | 91705 | 91730 | 91755 | 91780 |
| 1.23 | 0.91805 | 91830 | 91855 | 91879 | 91904 | 91929 | 91953 | 91978 | 92002 | 92026 |
| 1.24 | 0.92051 | 92075 | 92099 | 92123 | 92147 | 92171 | 92195 | 92219 | 92243 | 92266 |
| 1.25 | 0.92290 | 92314 | 92337 | 92361 | 92384 | 92408 | 92431 | 92454 | 92477 | 92500 |
| 1.26 | 0.92524 | 92547 | 92570 | 92593 | 92615 | 92638 | 92661 | 92684 | 92706 | 92729 |
| 1.27 | 0.92751 | 92774 | 92796 | 92819 | 92841 | 92863 | 92885 | 92907 | 92929 | 92951 |
| 1.28 | 0.92973 | 92995 | 93017 | 93039 | 93061 | 93082 | 93104 | 93126 | 93147 | 93168 |
| 1.29 | 0.93190 | 93211 | 93232 | 93254 | 93275 | 93296 | 93317 | 93338 | 93359 | 93380 |
| 1.30 | 0.93401 | 93422 | 93442 | 93463 | 93484 | 93504 | 93525 | 93545 | 93566 | 93586 |
| 1.31 | 0.93606 | 93627 | 93647 | 93667 | 93687 | 93707 | 93727 | 93747 | 93767 | 93787 |
| 1.32 | 0.93807 | 93826 | 93846 | 93866 | 93885 | 93905 | 93924 | 93944 | 93963 | 93982 |
| 1.33 | 0.94002 | 94021 | 94040 | 94059 | 94078 | 94097 | 94116 | 94135 | 94154 | 94173 |
| 1.34 | 0.94191 | 94210 | 94229 | 94247 | 94266 | 94284 | 94303 | 94321 | 94340 | 94358 |
| 1.35 | 0.94376 | 94394 | 94413 | 94431 | 94449 | 94467 | 94485 | 94503 | 94521 | 94538 |
| 1.36 | 0.94556 | 94574 | 94592 | 94609 | 94627 | 94644 | 94662 | 94679 | 94697 | 94714 |
| 1.37 | 0.94731 | 94748 | 94766 | 94783 | 94800 | 94817 | 94834 | 94851 | 94868 | 94885 |
| 1.38 | 0.94902 | 94918 | 94935 | 94952 | 94968 | 94985 | 95002 | 95018 | 95035 | 95051 |
| 1.39 | 0.95067 | 95084 | 95100 | 95116 | 95132 | 95148 | 95165 | 95181 | 95197 | 95213 |
| 1.40 | 0.95229 | 95244 | 95260 | 95276 | 95292 | 95307 | 95323 | 95339 | 95354 | 95370 |
| 1.41 | 0.95385 | 95401 | 95416 | 95431 | 95447 | 95462 | 95477 | 95492 | 95507 | 95523 |
| 1.42 | 0.95538 | 95553 | 95568 | 95582 | 95597 | 95612 | 95627 | 95642 | 95656 | 95671 |
| 1.43 | 0.95686 | 95700 | 95715 | 95729 | 95744 | 95758 | 95773 | 95787 | 95801 | 95815 |
| 1.44 | 0.95830 | 95844 | 95858 | 95872 | 95886 | 95900 | 95914 | 95928 | 95942 | 95956 |
| 1.45 | 0.95970 | 95983 | 95997 | 96011 | 96024 | 96038 | 96051 | 96065 | 96078 | 96092 |
| 1.46 | 0.96105 | 96119 | 96132 | 96145 | 96159 | 96172 | 96185 | 96198 | 96211 | 96224 |
| 1.47 | 0.96237 | 96250 | 96263 | 96276 | 96289 | 96302 | 96315 | 96327 | 96340 | 96353 |
| 1.48 | 0.96365 | 96378 | 96391 | 96403 | 96416 | 96428 | 96440 | 96453 | 96465 | 96478 |
| 1.49 | 0.96490 | 96502 | 96514 | 96526 | 96539 | 96551 | 96563 | 96575 | 96587 | 96599 |

The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx. \right)$$

| x | 0 | 2 | 4 | 6 | 8 | x | 0 | 2 | 4 | 6 | 8 |
|------|---------|-------|-------|-------|-------|------|---------|-------|-------|-------|-------|
| 1.50 | 0.96611 | 96634 | 96658 | 96681 | 96705 | 2.00 | 0.99532 | 99536 | 99540 | 99544 | 99548 |
| 1.51 | 0.96728 | 96751 | 96774 | 96796 | 96819 | 2.01 | 0.99552 | 99556 | 99560 | 99564 | 99568 |
| 1.52 | 0.96841 | 96864 | 96886 | 96908 | 96930 | 2.02 | 0.99572 | 99576 | 99580 | 99583 | 99587 |
| 1.53 | 0.96952 | 96973 | 96995 | 97016 | 97037 | 2.03 | 0.99591 | 99594 | 99598 | 99601 | 99605 |
| 1.54 | 0.97059 | 97080 | 97100 | 97121 | 97142 | 2.04 | 0.99609 | 99612 | 99616 | 99619 | 99622 |
| 1.55 | 0.97162 | 97183 | 97203 | 97223 | 97243 | 2.05 | 0.99626 | 99629 | 99633 | 99636 | 99639 |
| 1.56 | 0.97263 | 97283 | 97302 | 97322 | 97341 | 2.06 | 0.99642 | 99646 | 99649 | 99652 | 99655 |
| 1.57 | 0.97360 | 97379 | 97398 | 97417 | 97436 | 2.07 | 0.99658 | 99661 | 99664 | 99667 | 99670 |
| 1.58 | 0.97455 | 97473 | 97492 | 97510 | 97528 | 2.08 | 0.99673 | 99676 | 99679 | 99682 | 99685 |
| 1.59 | 0.97546 | 97564 | 97582 | 97600 | 97617 | 2.09 | 0.99688 | 99691 | 99694 | 99697 | 99699 |
| 1.60 | 0.97635 | 97652 | 97670 | 97687 | 97704 | 2.10 | 0.99702 | 99705 | 99707 | 99710 | 99713 |
| 1.61 | 0.97721 | 97738 | 97754 | 97771 | 97787 | 2.11 | 0.99715 | 99718 | 99721 | 99723 | 99726 |
| 1.62 | 0.97804 | 97820 | 97836 | 97852 | 97868 | 2.12 | 0.99728 | 99731 | 99733 | 99736 | 99738 |
| 1.63 | 0.97884 | 97900 | 97916 | 97931 | 97947 | 2.13 | 0.99741 | 99743 | 99745 | 99748 | 99750 |
| 1.64 | 0.97962 | 97977 | 97993 | 98008 | 98023 | 2.14 | 0.99753 | 99755 | 99757 | 99759 | 99762 |
| 1.65 | 0.98038 | 98052 | 98067 | 98082 | 98096 | 2.15 | 0.99764 | 99766 | 99768 | 99770 | 99773 |
| 1.66 | 0.98110 | 98125 | 98139 | 98153 | 98167 | 2.16 | 0.99775 | 99777 | 99779 | 99781 | 99783 |
| 1.67 | 0.98181 | 98195 | 98209 | 98222 | 98236 | 2.17 | 0.99785 | 99787 | 99789 | 99791 | 99793 |
| 1.68 | 0.98249 | 98263 | 98276 | 98289 | 98302 | 2.18 | 0.99795 | 99797 | 99799 | 99801 | 99803 |
| 1.69 | 0.98315 | 98328 | 98341 | 98354 | 98366 | 2.19 | 0.99805 | 99806 | 99808 | 99810 | 99812 |
| 1.70 | 0.98379 | 98392 | 98404 | 98416 | 98429 | 2.20 | 0.99814 | 99815 | 99817 | 99819 | 99821 |
| 1.71 | 0.98441 | 98453 | 98465 | 98477 | 98489 | 2.21 | 0.99822 | 99824 | 99826 | 99827 | 99829 |
| 1.72 | 0.98500 | 98512 | 98524 | 98535 | 98546 | 2.22 | 0.99831 | 99832 | 99834 | 99836 | 99837 |
| 1.73 | 0.98558 | 98569 | 98580 | 98591 | 98602 | 2.23 | 0.99839 | 99840 | 99842 | 99843 | 99845 |
| 1.74 | 0.98613 | 98624 | 98635 | 98646 | 98657 | 2.24 | 0.99846 | 99848 | 99849 | 99851 | 99852 |
| 1.75 | 0.98667 | 98678 | 98688 | 98699 | 98709 | 2.25 | 0.99854 | 99855 | 99857 | 99858 | 99859 |
| 1.76 | 0.98719 | 98729 | 98739 | 98749 | 98759 | 2.26 | 0.99861 | 99862 | 99863 | 99865 | 99866 |
| 1.77 | 0.98769 | 98779 | 98789 | 98798 | 98808 | 2.27 | 0.99867 | 99869 | 99870 | 99871 | 99873 |
| 1.78 | 0.98817 | 98827 | 98836 | 98846 | 98855 | 2.28 | 0.99874 | 99875 | 99876 | 99877 | 99879 |
| 1.79 | 0.98864 | 98873 | 98882 | 98891 | 98900 | 2.29 | 0.99880 | 99881 | 99882 | 99883 | 99885 |
| 1.80 | 0.98909 | 98918 | 98927 | 98935 | 98944 | 2.30 | 0.99886 | 99887 | 99888 | 99889 | 99890 |
| 1.81 | 0.98952 | 98961 | 98969 | 98978 | 98986 | 2.31 | 0.99891 | 99892 | 99893 | 99894 | 99896 |
| 1.82 | 0.98994 | 99003 | 99011 | 99019 | 99027 | 2.32 | 0.99897 | 99898 | 99899 | 99900 | 99901 |
| 1.83 | 0.99035 | 99043 | 99050 | 99058 | 99066 | 2.33 | 0.99902 | 99903 | 99904 | 99905 | 99906 |
| 1.84 | 0.99074 | 99081 | 99089 | 99096 | 99104 | 2.34 | 0.99906 | 99907 | 99908 | 99909 | 99910 |
| 1.85 | 0.99111 | 99118 | 99126 | 99133 | 99140 | 2.35 | 0.99911 | 99912 | 99913 | 99914 | 99915 |
| 1.86 | 0.99147 | 99154 | 99161 | 99168 | 99175 | 2.36 | 0.99915 | 99916 | 99917 | 99918 | 99919 |
| 1.87 | 0.99182 | 99189 | 99196 | 99202 | 99209 | 2.37 | 0.99920 | 99920 | 99921 | 99922 | 99923 |
| 1.88 | 0.99216 | 99222 | 99229 | 99235 | 99242 | 2.38 | 0.99924 | 99924 | 99925 | 99926 | 99927 |
| 1.89 | 0.99248 | 99254 | 99261 | 99267 | 99273 | 2.39 | 0.99928 | 99928 | 99929 | 99930 | 99930 |
| 1.90 | 0.99279 | 99285 | 99291 | 99297 | 99303 | 2.40 | 0.99931 | 99932 | 99933 | 99933 | 99934 |
| 1.91 | 0.99309 | 99315 | 99321 | 99326 | 99332 | 2.41 | 0.99935 | 99935 | 99936 | 99937 | 99937 |
| 1.92 | 0.99338 | 99343 | 99349 | 99355 | 99360 | 2.42 | 0.99938 | 99939 | 99939 | 99940 | 99940 |
| 1.93 | 0.99366 | 99371 | 99376 | 99382 | 99387 | 2.43 | 0.99941 | 99942 | 99942 | 99943 | 99943 |
| 1.94 | 0.99392 | 99397 | 99403 | 99408 | 99413 | 2.44 | 0.99944 | 99945 | 99945 | 99946 | 99946 |
| 1.95 | 0.99418 | 99423 | 99428 | 99433 | 99438 | 2.45 | 0.99947 | 99947 | 99948 | 99949 | 99949 |
| 1.96 | 0.99443 | 99447 | 99452 | 99457 | 99462 | 2.46 | 0.99950 | 99950 | 99951 | 99951 | 99952 |
| 1.97 | 0.99466 | 99471 | 99476 | 99480 | 99485 | 2.47 | 0.99952 | 99953 | 99953 | 99954 | 99954 |
| 1.98 | 0.99489 | 99494 | 99498 | 99502 | 99507 | 2.48 | 0.99955 | 99955 | 99956 | 99956 | 99957 |
| 1.99 | 0.99511 | 99515 | 99520 | 99524 | 99528 | 2.49 | 0.99957 | 99958 | 99958 | 99958 | 99959 |
| 2.00 | 0.99532 | 99536 | 99540 | 99544 | 99548 | 2.50 | 0.99959 | 99960 | 99960 | 99961 | 99961 |

The Probability Integral.

$$\left(\frac{2}{\sqrt{\pi}}\int_0^xe^{-x^2}dx.\right)$$

| <i>x</i> | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----------|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2.5 | 0.99959 | 99961 | 99963 | 99965 | 99967 | 99969 | 99971 | 99972 | 99974 | 99975 |
| 2.6 | 0.99976 | 99978 | 99979 | 99980 | 99981 | 99982 | 99983 | 99984 | 99985 | 99986 |
| 2.7 | 0.99987 | 99987 | 99988 | 99989 | 99989 | 99990 | 99991 | 99991 | 99992 | 99992 |
| 2.8 | 0.99992 | 99993 | 99993 | 99994 | 99994 | 99994 | 99995 | 99995 | 99995 | 99996 |
| 2.9 | 0.99996 | 99996 | 99996 | 99997 | 99997 | 99997 | 99997 | 99997 | 99997 | 99998 |
| 3.0 | 0.99998 | 99998 | 99998 | 99998 | 99998 | 99998 | 99998 | 99998 | 99999 | 99999 |

The value, *I*, of the Probability Integral may always be found from the convergent series

$$I=\frac{2}{\sqrt{\pi}}\left(x-\frac{x^3}{3\cdot1!}+\frac{x^5}{5\cdot2!}-\frac{x^7}{7\cdot3!}+\cdots\right),$$

but for large values of *x*, the semiconvergent series

$$I=1-\frac{e^{-x}}{x\sqrt{\pi}}\left(1-\frac{1}{2x^2}+\frac{1\cdot3}{(2x^2)^2}-\frac{1\cdot3\cdot5}{(2x^2)^3}+\cdots\right)$$

is convenient.

Values of the Complete Elliptic Integrals, K and E , for Different Values of the Modulus, k .

$$K = \int_0^{\frac{\pi}{2}} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}; E = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 z} \cdot dz.$$

| $\sin^{-1} k$ | K | E | $\sin^{-1} k$ | K | E | $\sin^{-1} k$ | K | E |
|---------------|--------|--------|---------------|--------|--------|---------------|--------|--------|
| 0° | 1.5708 | 1.5708 | 30° | 1.6858 | 1.4675 | 60° | 2.1565 | 1.2111 |
| 1° | 1.5709 | 1.5707 | 31° | 1.6941 | 1.4608 | 61° | 2.1842 | 1.2015 |
| 2° | 1.5713 | 1.5703 | 32° | 1.7028 | 1.4539 | 62° | 2.2132 | 1.1920 |
| 3° | 1.5719 | 1.5697 | 33° | 1.7119 | 1.4469 | 63° | 2.2435 | 1.1826 |
| 4° | 1.5727 | 1.5689 | 34° | 1.7214 | 1.4397 | 64° | 2.2754 | 1.1732 |
| 5° | 1.5738 | 1.5678 | 35° | 1.7312 | 1.4323 | 65° | 2.3088 | 1.1638 |
| 6° | 1.5751 | 1.5665 | 36° | 1.7415 | 1.4248 | 66° | 2.3439 | 1.1545 |
| 7° | 1.5767 | 1.5649 | 37° | 1.7522 | 1.4171 | 67° | 2.3809 | 1.1453 |
| 8° | 1.5785 | 1.5632 | 38° | 1.7633 | 1.4092 | 68° | 2.4198 | 1.1362 |
| 9° | 1.5805 | 1.5611 | 39° | 1.7748 | 1.4013 | 69° | 2.4610 | 1.1272 |
| 10° | 1.5828 | 1.5589 | 40° | 1.7868 | 1.3931 | 70° | 2.5046 | 1.1184 |
| 11° | 1.5854 | 1.5564 | 41° | 1.7992 | 1.3849 | 71° | 2.5507 | 1.1096 |
| 12° | 1.5882 | 1.5537 | 42° | 1.8122 | 1.3765 | 72° | 2.5998 | 1.1011 |
| 13° | 1.5913 | 1.5507 | 43° | 1.8256 | 1.3680 | 73° | 2.6521 | 1.0927 |
| 14° | 1.5946 | 1.5476 | 44° | 1.8396 | 1.3594 | 74° | 2.7081 | 1.0844 |
| 15° | 1.5981 | 1.5442 | 45° | 1.8541 | 1.3506 | 75° | 2.7681 | 1.0764 |
| 16° | 1.6020 | 1.5405 | 46° | 1.8691 | 1.3418 | 76° | 2.8327 | 1.0686 |
| 17° | 1.6061 | 1.5367 | 47° | 1.8848 | 1.3329 | 77° | 2.9026 | 1.0611 |
| 18° | 1.6105 | 1.5326 | 48° | 1.9011 | 1.3238 | 78° | 2.9786 | 1.0538 |
| 19° | 1.6151 | 1.5283 | 49° | 1.9180 | 1.3147 | 79° | 3.0617 | 1.0468 |
| 20° | 1.6200 | 1.5238 | 50° | 1.9356 | 1.3055 | 80° | 3.1534 | 1.0401 |
| 21° | 1.6252 | 1.5191 | 51° | 1.9539 | 1.2963 | 81° | 3.2553 | 1.0338 |
| 22° | 1.6307 | 1.5141 | 52° | 1.9729 | 1.2870 | 82° | 3.3699 | 1.0278 |
| 23° | 1.6365 | 1.5090 | 53° | 1.9927 | 1.2776 | 83° | 3.5004 | 1.0223 |
| 24° | 1.6426 | 1.5037 | 54° | 2.0133 | 1.2681 | 84° | 3.6519 | 1.0172 |
| 25° | 1.6490 | 1.4981 | 55° | 2.0347 | 1.2587 | 85° | 3.8317 | 1.0127 |
| 26° | 1.6557 | 1.4924 | 56° | 2.0571 | 1.2492 | 86° | 4.0528 | 1.0086 |
| 27° | 1.6627 | 1.4864 | 57° | 2.0804 | 1.2397 | 87° | 4.3387 | 1.0053 |
| 28° | 1.6701 | 1.4803 | 58° | 2.1047 | 1.2301 | 88° | 4.7427 | 1.0026 |
| 29° | 1.6777 | 1.4740 | 59° | 2.1300 | 1.2206 | 89° | 5.4349 | 1.0008 |

Values of $F(k, \phi)$ for Certain Values of k and ϕ .

$$F(k, \phi) = \int_0^\phi \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}.$$

| ϕ | $\alpha = \sin^{-1} k.$ | | | | | | | | |
|--------|-------------------------|--------|--------|--------|--------|--------|--------|--------|--------|
| | 0° | 10° | 15° | 30° | 45° | 60° | 75° | 80° | 90° |
| 1° | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 |
| 2° | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 |
| 3° | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 |
| 4° | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0699 | 0.0699 | 0.0699 | 0.0699 |
| 5° | 0.0873 | 0.0873 | 0.0873 | 0.0873 | 0.0873 | 0.0874 | 0.0874 | 0.0874 | 0.0874 |
| 10° | 0.1745 | 0.1746 | 0.1746 | 0.1748 | 0.1750 | 0.1752 | 0.1754 | 0.1754 | 0.1754 |
| 15° | 0.2618 | 0.2619 | 0.2620 | 0.2625 | 0.2633 | 0.2641 | 0.2646 | 0.2647 | 0.2648 |
| 20° | 0.3491 | 0.3493 | 0.3495 | 0.3508 | 0.3526 | 0.3545 | 0.3559 | 0.3562 | 0.3564 |
| 25° | 0.4363 | 0.4367 | 0.4372 | 0.4397 | 0.4433 | 0.4470 | 0.4498 | 0.4504 | 0.4509 |
| 30° | 0.5236 | 0.5243 | 0.5251 | 0.5294 | 0.5356 | 0.5422 | 0.5474 | 0.5484 | 0.5493 |
| 35° | 0.6109 | 0.6119 | 0.6132 | 0.6200 | 0.6300 | 0.6408 | 0.6495 | 0.6513 | 0.6528 |
| 40° | 0.6981 | 0.6997 | 0.7016 | 0.7116 | 0.7267 | 0.7436 | 0.7574 | 0.7604 | 0.7629 |
| 45° | 0.7854 | 0.7876 | 0.7902 | 0.8044 | 0.8260 | 0.8512 | 0.8727 | 0.8774 | 0.8814 |
| 50° | 0.8727 | 0.8756 | 0.8792 | 0.8982 | 0.9283 | 0.9646 | 0.9971 | 1.0044 | 1.0107 |
| 55° | 0.9599 | 0.9637 | 0.9683 | 0.9933 | 1.0337 | 1.0848 | 1.1331 | 1.1444 | 1.1542 |
| 60° | 1.0472 | 1.0519 | 1.0577 | 1.0896 | 1.1424 | 1.2125 | 1.2837 | 1.3014 | 1.3170 |
| 65° | 1.1345 | 1.1402 | 1.1474 | 1.1869 | 1.2545 | 1.3489 | 1.4532 | 1.4810 | 1.5064 |
| 70° | 1.2217 | 1.2286 | 1.2373 | 1.2853 | 1.3697 | 1.4944 | 1.6468 | 1.6918 | 1.7354 |
| 75° | 1.3090 | 1.3171 | 1.3273 | 1.3846 | 1.4879 | 1.6492 | 1.8714 | 1.9468 | 2.0276 |
| 80° | 1.3963 | 1.4056 | 1.4175 | 1.4846 | 1.6085 | 1.8125 | 2.1339 | 2.2653 | 2.4362 |
| 85° | 1.4835 | 1.4942 | 1.5078 | 1.5850 | 1.7308 | 1.9826 | 2.4366 | 2.6694 | 3.1313 |
| 86° | 1.5010 | 1.5120 | 1.5259 | 1.6052 | 1.7554 | 2.0172 | 2.5013 | 2.7612 | 3.3547 |
| 87° | 1.5184 | 1.5297 | 1.5439 | 1.6253 | 1.7801 | 2.0519 | 2.5670 | 2.8561 | 3.6425 |
| 88° | 1.5359 | 1.5474 | 1.5620 | 1.6454 | 1.8047 | 2.0867 | 2.6336 | 2.9537 | 4.0481 |
| 89° | 1.5533 | 1.5651 | 1.5801 | 1.6656 | 1.8294 | 2.1216 | 2.7007 | 3.0530 | 4.7414 |
| 90° | 1.5708 | 1.5828 | 1.5981 | 1.6858 | 1.8541 | 2.1565 | 2.7681 | 3.1534 | Inf. |

Values of $E(k, \phi)$ for Certain Values of k and ϕ .

$$E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 z} \cdot dz.$$

| ϕ | $\alpha = \sin^{-1} k.$ | | | | | | | | |
|--------|-------------------------|--------|--------|--------|--------|--------|--------|--------|--------|
| | 0° | 10° | 15° | 30° | 45° | 60° | 75° | 80° | 90° |
| 1° | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 | 0.0174 |
| 2° | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 | 0.0349 |
| 3° | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0524 | 0.0523 | 0.0523 | 0.0523 | 0.0523 |
| 4° | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 | 0.0698 |
| 5° | 0.0873 | 0.0873 | 0.0873 | 0.0872 | 0.0872 | 0.0872 | 0.0872 | 0.0872 | 0.0872 |
| 10° | 0.1745 | 0.1745 | 0.1745 | 0.1743 | 0.1741 | 0.1739 | 0.1737 | 0.1737 | 0.1736 |
| 15° | 0.2618 | 0.2617 | 0.2616 | 0.2611 | 0.2603 | 0.2596 | 0.2590 | 0.2589 | 0.2588 |
| 20° | 0.3491 | 0.3489 | 0.3486 | 0.3473 | 0.3456 | 0.3438 | 0.3425 | 0.3422 | 0.3420 |
| 25° | 0.4363 | 0.4359 | 0.4354 | 0.4330 | 0.4296 | 0.4261 | 0.4236 | 0.4230 | 0.4226 |
| 30° | 0.5236 | 0.5229 | 0.5221 | 0.5179 | 0.5120 | 0.5061 | 0.5016 | 0.5007 | 0.5000 |
| 35° | 0.6109 | 0.6098 | 0.6085 | 0.6019 | 0.5928 | 0.5833 | 0.5762 | 0.5748 | 0.5736 |
| 40° | 0.6981 | 0.6966 | 0.6947 | 0.6851 | 0.6715 | 0.6575 | 0.6468 | 0.6446 | 0.6428 |
| 45° | 0.7854 | 0.7832 | 0.7806 | 0.7672 | 0.7482 | 0.7282 | 0.7129 | 0.7097 | 0.7071 |
| 50° | 0.8727 | 0.8698 | 0.8663 | 0.8483 | 0.8226 | 0.7954 | 0.7741 | 0.7697 | 0.7660 |
| 55° | 0.9599 | 0.9562 | 0.9517 | 0.9284 | 0.8949 | 0.8588 | 0.8302 | 0.8242 | 0.8192 |
| 60° | 1.0472 | 1.0426 | 1.0368 | 1.0076 | 0.9650 | 0.9184 | 0.8808 | 0.8728 | 0.8660 |
| 65° | 1.1345 | 1.1288 | 1.1218 | 1.0858 | 1.0329 | 0.9743 | 0.9258 | 0.9152 | 0.9063 |
| 70° | 1.2217 | 1.2149 | 1.2065 | 1.1632 | 1.0990 | 1.0266 | 0.9652 | 0.9514 | 0.9397 |
| 75° | 1.3090 | 1.3010 | 1.2911 | 1.2399 | 1.1635 | 1.0759 | 0.9992 | 0.9814 | 0.9659 |
| 80° | 1.3963 | 1.3870 | 1.3755 | 1.3161 | 1.2266 | 1.1225 | 1.0282 | 1.0054 | 0.9848 |
| 85° | 1.4835 | 1.4729 | 1.4598 | 1.3919 | 1.2889 | 1.1673 | 1.0534 | 1.0244 | 0.9962 |
| 86° | 1.5010 | 1.4901 | 1.4767 | 1.4070 | 1.3012 | 1.1761 | 1.0581 | 1.0277 | 0.9976 |
| 87° | 1.5184 | 1.5073 | 1.4936 | 1.4221 | 1.3136 | 1.1848 | 1.0628 | 1.0309 | 0.9986 |
| 88° | 1.5359 | 1.5245 | 1.5104 | 1.4372 | 1.3260 | 1.1936 | 1.0674 | 1.0340 | 0.9994 |
| 89° | 1.5533 | 1.5417 | 1.5273 | 1.4524 | 1.3383 | 1.2023 | 1.0719 | 1.0371 | 0.9998 |
| 90° | 1.5708 | 1.5589 | 1.5442 | 1.4675 | 1.3506 | 1.2111 | 1.0764 | 1.0401 | 1.0000 |

Hyperbolic Functions.

| x | e^x | e^{-x} | $\sinh x$ | $\cosh x$ | $\operatorname{gd} x$ |
|------|--------|----------|-----------|-----------|-----------------------|
| 0.00 | 1.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 |
| .01 | 1.0100 | 0.9900 | .0100 | 1.0000 | 0.5729 |
| .02 | 1.0202 | .9802 | .0200 | 1.0002 | 1.1458 |
| .03 | 1.0305 | .9704 | .0300 | 1.0004 | 1.7186 |
| .04 | 1.0408 | .9608 | .0400 | 1.0008 | 2.2912 |
| .05 | 1.0513 | .9512 | .0500 | 1.0013 | 2.8636 |
| .06 | 1.0618 | .9418 | .0600 | 1.0018 | 3.4357 |
| .07 | 1.0725 | .9324 | .0701 | 1.0025 | 4.0074 |
| .08 | 1.0833 | .9231 | .0801 | 1.0032 | 4.5788 |
| .09 | 1.0942 | .9139 | .0901 | 1.0041 | 5.1497 |
| .10 | 1.1052 | .9048 | .1002 | 1.0050 | 5.720 |
| .11 | 1.1163 | .8958 | .1102 | 1.0061 | 6.290 |
| .12 | 1.1275 | .8869 | .1203 | 1.0072 | 6.859 |
| .13 | 1.1388 | .8781 | .1304 | 1.0085 | 7.428 |
| .14 | 1.1503 | .8694 | .1405 | 1.0098 | 7.995 |
| .15 | 1.1618 | .8607 | .1506 | 1.0113 | 8.562 |
| .16 | 1.1735 | .8521 | .1607 | 1.0128 | 9.128 |
| .17 | 1.1853 | .8437 | .1708 | 1.0145 | 9.694 |
| .18 | 1.1972 | .8353 | .1810 | 1.0162 | 10.258 |
| .19 | 1.2092 | .8270 | .1911 | 1.0181 | 10.821 |
| .20 | 1.2214 | .8187 | .2013 | 1.0201 | 11.384 |
| .21 | 1.2337 | .8106 | .2115 | 1.0221 | 11.945 |
| .22 | 1.2461 | .8025 | .2218 | 1.0243 | 12.505 |
| .23 | 1.2586 | .7945 | .2320 | 1.0266 | 13.063 |
| .24 | 1.2712 | .7866 | .2423 | 1.0289 | 13.621 |
| .25 | 1.2840 | .7788 | .2526 | 1.0314 | 14.177 |
| .26 | 1.2969 | .7711 | .2629 | 1.0340 | 14.732 |
| .27 | 1.3100 | .7634 | .2733 | 1.0367 | 15.285 |
| .28 | 1.3231 | .7558 | .2837 | 1.0395 | 15.837 |
| .29 | 1.3364 | .7483 | .2941 | 1.0423 | 16.388 |
| .30 | 1.3499 | .7408 | .3045 | 1.0453 | 16.937 |
| .31 | 1.3634 | .7334 | .3150 | 1.0484 | 17.484 |
| .32 | 1.3771 | .7261 | .3255 | 1.0516 | 18.030 |
| .33 | 1.3910 | .7189 | .3360 | 1.0549 | 18.573 |
| .34 | 1.4049 | .7118 | .3466 | 1.0584 | 19.116 |
| .35 | 1.4191 | .7047 | .3572 | 1.0619 | 19.656 |
| .36 | 1.4333 | .6977 | .3678 | 1.0655 | 20.195 |
| .37 | 1.4477 | .6907 | .3785 | 1.0692 | 20.732 |
| .38 | 1.4623 | .6839 | .3892 | 1.0731 | 21.267 |
| .39 | 1.4770 | .6771 | .4000 | 1.0770 | 21.800 |
| .40 | 1.4918 | .6703 | .4108 | 1.0811 | 22.331 |
| .41 | 1.5068 | .6636 | .4216 | 1.0852 | 22.859 |
| .42 | 1.5220 | .6570 | .4325 | 1.0895 | 23.386 |
| .43 | 1.5373 | .6505 | .4434 | 1.0939 | 23.911 |
| .44 | 1.5527 | .6440 | .4543 | 1.0984 | 24.434 |
| .45 | 1.5683 | .6376 | .4653 | 1.1030 | 24.955 |
| .46 | 1.5841 | .6313 | .4764 | 1.1077 | 25.473 |
| .47 | 1.6000 | .6250 | .4875 | 1.1125 | 25.989 |
| .48 | 1.6161 | .6188 | .4986 | 1.1174 | 26.503 |
| .49 | 1.6323 | .6126 | .5098 | 1.1225 | 27.015 |
| 0.50 | 1.6487 | 0.6065 | 0.5211 | 1.1276 | 27.524 |

NOTE.—This table is taken from Prof. Byerly's Treatise on Fourier's Series, published by Messrs. Ginn & Co.

Hyperbolic Functions.

| x | e^x | e^{-x} | $\sinh x$ | $\cosh x$ | $\operatorname{gd} x$ |
|------|--------|----------|-----------|-----------|-----------------------|
| 0.50 | 1.6487 | 0.6065 | 0.5211 | 1.1276 | 27.524 |
| .51 | 1.6653 | .6005 | .5324 | 1.1329 | 28.031 |
| .52 | 1.6820 | .5945 | .5438 | 1.1383 | 28.535 |
| .53 | 1.6989 | .5886 | .5552 | 1.1438 | 29.037 |
| .54 | 1.7160 | .5827 | .5666 | 1.1494 | 29.537 |
| .55 | 1.7333 | .5770 | .5782 | 1.1551 | 30.034 |
| .56 | 1.7507 | .5712 | .5897 | 1.1609 | 30.529 |
| .57 | 1.7683 | .5655 | .6014 | 1.1669 | 31.021 |
| .58 | 1.7860 | .5599 | .6131 | 1.1730 | 31.511 |
| .59 | 1.8040 | .5543 | .6248 | 1.1792 | 31.998 |
| .60 | 1.8221 | .5488 | .6367 | 1.1855 | 32.483 |
| .61 | 1.8404 | .5433 | .6485 | 1.1919 | 32.965 |
| .62 | 1.8589 | .5379 | .6605 | 1.1984 | 33.444 |
| .63 | 1.8776 | .5326 | .6725 | 1.2051 | 33.921 |
| .64 | 1.8965 | .5273 | .6846 | 1.2119 | 34.395 |
| .65 | 1.9155 | .5220 | .6967 | 1.2188 | 34.867 |
| .66 | 1.9348 | .5169 | .7090 | 1.2258 | 35.336 |
| .67 | 1.9542 | .5117 | .7213 | 1.2330 | 35.802 |
| .68 | 1.9739 | .5066 | .7336 | 1.2402 | 36.265 |
| .69 | 1.9937 | .5016 | .7461 | 1.2476 | 36.726 |
| .70 | 2.0138 | .4966 | .7586 | 1.2552 | 37.183 |
| .71 | 2.0340 | .4916 | .7712 | 1.2628 | 37.638 |
| .72 | 2.0544 | .4867 | .7838 | 1.2706 | 38.091 |
| .73 | 2.0751 | .4819 | .7966 | 1.2785 | 38.540 |
| .74 | 2.0959 | .4771 | .8094 | 1.2865 | 38.987 |
| .75 | 2.1170 | .4724 | .8223 | 1.2947 | 39.431 |
| .76 | 2.1383 | .4677 | .8353 | 1.3030 | 39.872 |
| .77 | 2.1598 | .4630 | .8484 | 1.3114 | 40.310 |
| .78 | 2.1815 | .4584 | .8615 | 1.3199 | 40.746 |
| .79 | 2.2034 | .4538 | .8748 | 1.3286 | 41.179 |
| .80 | 2.2255 | .4493 | .8881 | 1.3374 | 41.608 |
| .81 | 2.2479 | .4449 | .9015 | 1.3464 | 42.035 |
| .82 | 2.2705 | .4404 | .9150 | 1.3555 | 42.460 |
| .83 | 2.2933 | .4360 | .9286 | 1.3647 | 42.881 |
| .84 | 2.3164 | .4317 | .9423 | 1.3740 | 43.299 |
| .85 | 2.3396 | .4274 | .9561 | 1.3835 | 43.715 |
| .86 | 2.3632 | .4232 | .9700 | 1.3932 | 44.128 |
| .87 | 2.3869 | .4190 | .9840 | 1.4029 | 44.537 |
| .88 | 2.4109 | .4148 | .9981 | 1.4128 | 44.944 |
| .89 | 2.4351 | .4107 | 1.0122 | 1.4229 | 45.348 |
| .90 | 2.4596 | .4066 | 1.0265 | 1.4331 | 45.750 |
| .91 | 2.4843 | .4025 | 1.0409 | 1.4434 | 46.148 |
| .92 | 2.5093 | .3985 | 1.0554 | 1.4539 | 46.544 |
| .93 | 2.5345 | .3946 | 1.0700 | 1.4645 | 46.936 |
| .94 | 2.5600 | .3906 | 1.0847 | 1.4753 | 47.326 |
| .95 | 2.5857 | .3867 | 1.0995 | 1.4862 | 47.713 |
| .96 | 2.6117 | .3829 | 1.1144 | 1.4973 | 48.097 |
| .97 | 2.6379 | .3791 | 1.1294 | 1.5085 | 48.478 |
| .98 | 2.6645 | .3753 | 1.1446 | 1.5199 | 48.857 |
| .99 | 2.6912 | .3716 | 1.1598 | 1.5314 | 49.232 |
| 1.00 | 2.7183 | 0.3679 | 1.1752 | 1.5431 | 49.605 |

$$\sinh x = \tan \operatorname{gd} x; \quad \cosh x = \sec \operatorname{gd} x; \quad \tanh x = \sin \operatorname{gd} x.$$

Hyperbolic Functions.

| x | $\ell \sinh x$ | $\ell \cosh x$ | x | $\ell \sinh x$ | $\ell \cosh x$ | x | $\ell \sinh x$ | $\ell \cosh x$ |
|------|----------------|----------------|------|----------------|----------------|------|----------------|----------------|
| 1.00 | 0.0701 | 0.1884 | 1.50 | 0.3282 | 0.3715 | 2.00 | 0.5595 | 0.5754 |
| 1.01 | .0758 | .1917 | 1.51 | .3330 | .3754 | 2.01 | .5640 | .5796 |
| 1.02 | .0815 | .1950 | 1.52 | .3378 | .3794 | 2.02 | .5685 | .5838 |
| 1.03 | .0871 | .1984 | 1.53 | .3426 | .3833 | 2.03 | .5730 | .5880 |
| 1.04 | .0927 | .2018 | 1.54 | .3474 | .3873 | 2.04 | .5775 | .5922 |
| 1.05 | .0982 | .2051 | 1.55 | .3521 | .3913 | 2.05 | .5820 | .5964 |
| 1.06 | .1038 | .2086 | 1.56 | .3569 | .3952 | 2.06 | .5865 | .6006 |
| 1.07 | .1093 | .2120 | 1.57 | .3616 | .3992 | 2.07 | .5910 | .6048 |
| 1.08 | .1148 | .2154 | 1.58 | .3663 | .4032 | 2.08 | .5955 | .6090 |
| 1.09 | .1203 | .2189 | 1.59 | .3711 | .4072 | 2.09 | .6000 | .6132 |
| 1.10 | .1257 | .2223 | 1.60 | .3758 | .4112 | 2.10 | .6044 | .6175 |
| 1.11 | .1311 | .2258 | 1.61 | .3805 | .4152 | 2.11 | .6089 | .6217 |
| 1.12 | .1365 | .2293 | 1.62 | .3852 | .4192 | 2.12 | .6134 | .6259 |
| 1.13 | .1419 | .2328 | 1.63 | .3899 | .4232 | 2.13 | .6178 | .6301 |
| 1.14 | .1472 | .2364 | 1.64 | .3946 | .4273 | 2.14 | .6223 | .6343 |
| 1.15 | .1525 | .2399 | 1.65 | .3992 | .4313 | 2.15 | .6268 | .6386 |
| 1.16 | .1578 | .2435 | 1.66 | .4039 | .4353 | 2.16 | .6312 | .6428 |
| 1.17 | .1631 | .2470 | 1.67 | .4086 | .4394 | 2.17 | .6357 | .6470 |
| 1.18 | .1684 | .2506 | 1.68 | .4132 | .4434 | 2.18 | .6401 | .6512 |
| 1.19 | .1736 | .2542 | 1.69 | .4179 | .4475 | 2.19 | .6446 | .6555 |
| 1.20 | .1788 | .2578 | 1.70 | .4225 | .4515 | 2.20 | .6491 | .6597 |
| 1.21 | .1840 | .2615 | 1.71 | .4272 | .4556 | 2.21 | .6535 | .6640 |
| 1.22 | .1892 | .2651 | 1.72 | .4318 | .4597 | 2.22 | .6580 | .6682 |
| 1.23 | .1944 | .2688 | 1.73 | .4364 | .4637 | 2.23 | .6624 | .6724 |
| 1.24 | .1995 | .2724 | 1.74 | .4411 | .4678 | 2.24 | .6668 | .6767 |
| 1.25 | .2046 | .2761 | 1.75 | .4457 | .4719 | 2.25 | .6713 | .6809 |
| 1.26 | .2098 | .2798 | 1.76 | .4503 | .4760 | 2.26 | .6757 | .6852 |
| 1.27 | .2148 | .2835 | 1.77 | .4549 | .4801 | 2.27 | .6802 | .6894 |
| 1.28 | .2199 | .2872 | 1.78 | .4595 | .4842 | 2.28 | .6846 | .6937 |
| 1.29 | .2250 | .2909 | 1.79 | .4641 | .4883 | 2.29 | .6890 | .6979 |
| 1.30 | .2300 | .2947 | 1.80 | .4687 | .4924 | 2.30 | .6935 | .7022 |
| 1.31 | .2351 | .2984 | 1.81 | .4733 | .4965 | 2.31 | .6979 | .7064 |
| 1.32 | .2401 | .3022 | 1.82 | .4778 | .5006 | 2.32 | .7023 | .7107 |
| 1.33 | .2451 | .3059 | 1.83 | .4824 | .5048 | 2.33 | .7067 | .7150 |
| 1.34 | .2501 | .3097 | 1.84 | .4870 | .5089 | 2.34 | .7112 | .7192 |
| 1.35 | .2551 | .3135 | 1.85 | .4915 | .5130 | 2.35 | .7156 | .7235 |
| 1.36 | .2600 | .3173 | 1.86 | .4961 | .5172 | 2.36 | .7200 | .7278 |
| 1.37 | .2650 | .3211 | 1.87 | .5007 | .5213 | 2.37 | .7244 | .7320 |
| 1.38 | .2699 | .3249 | 1.88 | .5052 | .5254 | 2.38 | .7289 | .7363 |
| 1.39 | .2748 | .3288 | 1.89 | .5098 | .5296 | 2.38 | .7333 | .7406 |
| 1.40 | .2797 | .3326 | 1.90 | .5143 | .5337 | 2.40 | .7377 | .7448 |
| 1.41 | .2846 | .3365 | 1.91 | .5188 | .5379 | 2.41 | .7421 | .7491 |
| 1.42 | .2895 | .3403 | 1.92 | .5234 | .5421 | 2.42 | .7465 | .7534 |
| 1.43 | .2944 | .3442 | 1.93 | .5279 | .5462 | 2.43 | .7509 | .7577 |
| 1.44 | .2993 | .3481 | 1.94 | .5324 | .5504 | 2.44 | .7553 | .7619 |
| 1.45 | .3041 | .3520 | 1.95 | .5370 | .5545 | 2.45 | .7597 | .7662 |
| 1.46 | .3090 | .3559 | 1.96 | .5415 | .5587 | 2.46 | .7642 | .7705 |
| 1.47 | .3138 | .3598 | 1.97 | .5460 | .5629 | 2.47 | .7686 | .7748 |
| 1.48 | .3186 | .3637 | 1.98 | .5505 | .5671 | 2.48 | .7730 | .7791 |
| 1.49 | .3234 | .3676 | 1.99 | .5550 | .5713 | 2.49 | .7774 | .7833 |
| 1.50 | 0.3282 | 0.3715 | 2.00 | 0.5595 | 0.5754 | 2.50 | 0.7818 | 0.7876 |

Hyperbolic Functions.

| x | $\text{lsinh } x$ | $\text{lcosh } x$ | x | $\text{lsinh } x$ | $\text{lcosh } x$ | x | $\text{lsinh } x$ | $\text{lcosh } x$ |
|------|-------------------|-------------------|------|-------------------|-------------------|------|-------------------|-------------------|
| 2.50 | 0.7818 | 0.7876 | 2.75 | 0.8915 | 0.8951 | 3.0 | 1.0008 | 1.0029 |
| 2.51 | .7862 | .7919 | 2.76 | .8959 | .8994 | 3.1 | 1.0444 | 1.0462 |
| 2.52 | .7906 | .7962 | 2.77 | .9003 | .9037 | 3.2 | 1.0880 | 1.0894 |
| 2.53 | .7950 | .8005 | 2.78 | .9046 | .9080 | 3.3 | 1.1316 | 1.1327 |
| 2.54 | .7994 | .8048 | 2.79 | .9090 | .9123 | 3.4 | 1.1751 | 1.1761 |
| 2.55 | .8038 | .8091 | 2.80 | .9134 | .9166 | 3.5 | 1.2186 | 1.2194 |
| 2.56 | .8082 | .8134 | 2.81 | .9178 | .9209 | 3.6 | 1.2621 | 1.2628 |
| 2.57 | .8126 | .8176 | 2.82 | .9221 | .9252 | 3.7 | 1.3056 | 1.3061 |
| 2.58 | .8169 | .8219 | 2.83 | .9265 | .9295 | 3.8 | 1.3491 | 1.3495 |
| 2.59 | .8213 | .8262 | 2.84 | .9309 | .9338 | 3.9 | 1.3925 | 1.3929 |
| 2.60 | .8257 | .8305 | 2.85 | .9353 | .9382 | 4.0 | 1.4360 | 1.4363 |
| 2.61 | .8301 | .8348 | 2.86 | .9396 | .9425 | 4.1 | 1.4795 | 1.4797 |
| 2.62 | .8345 | .8391 | 2.87 | .9440 | .9468 | 4.2 | 1.5229 | 1.5231 |
| 2.63 | .8389 | .8434 | 2.88 | .9484 | .9511 | 4.3 | 1.5664 | 1.5665 |
| 2.64 | .8433 | .8477 | 2.89 | .9527 | .9554 | 4.4 | 1.6098 | 1.6099 |
| 2.65 | .8477 | .8520 | 2.90 | .9571 | .9597 | 4.5 | 1.6532 | 1.6533 |
| 2.66 | .8521 | .8563 | 2.91 | .9615 | .9641 | 4.6 | 1.6967 | 1.6968 |
| 2.67 | .8564 | .8606 | 2.92 | .9658 | .9684 | 4.7 | 1.7401 | 1.7402 |
| 2.68 | .8608 | .8649 | 2.93 | .9702 | .9727 | 4.8 | 1.7836 | 1.7836 |
| 2.69 | .8652 | .8692 | 2.94 | .9746 | .9770 | 4.9 | 1.8270 | 1.8270 |
| 2.70 | .8696 | .8735 | 2.95 | .9789 | .9813 | 5.0 | 1.8704 | 1.8705 |
| 2.71 | .8740 | .8778 | 2.96 | .9833 | .9856 | 6.0 | 2.3047 | 2.3047 |
| 2.72 | .8784 | .8821 | 2.97 | .9877 | .9900 | 7.0 | 2.7390 | 2.7390 |
| 2.73 | .8827 | .8864 | 2.98 | .9920 | .9943 | 8.0 | 3.1733 | 3.1733 |
| 2.74 | .8871 | .8907 | 2.99 | .9964 | .9986 | 9.0 | 3.6076 | 3.6076 |
| 2.75 | 0.8915 | 0.8951 | 3.00 | 1.0008 | 1.0029 | 10.0 | 4.0419 | 4.0419 |

For values of x greater than 7.0, we may write, to five places of decimals at least,

$$\log_{10} \sinh x = \log_{10} \cosh x = \log \frac{1}{2} e^x = x (0.4342945) + \bar{1}.6989700.$$

The Values of e^{-x} for Certain Values of x .

| x | e^{-x} | x | e^{-x} | x | e^{-x} | x | e^{-x} |
|------|----------|-------|----------|-------|----------|------|----------|
| 1/10 | 0.90484 | 8/10 | 0.44933 | 18/10 | 0.16530 | 5 | 0.00674 |
| 1/8 | 0.88250 | 9/10 | 0.40657 | 2 | 0.13534 | 11/2 | 0.00409 |
| 1/6 | 0.84648 | 1 | 0.36788 | 9/4 | 0.10540 | 6 | 0.00248 |
| 2/10 | 0.81873 | 11/10 | 0.33287 | 5/2 | 0.08209 | 13/2 | 0.00150 |
| 1/4 | 0.77880 | 9/8 | 0.32465 | 8/3 | 0.06948 | 7 | 0.00091 |
| 3/10 | 0.74082 | 12/10 | 0.30119 | 3 | 0.04979 | 15/2 | 0.00055 |
| 1/3 | 0.71653 | 5/4 | 0.28650 | 25/8 | 0.04394 | 8 | 0.00034 |
| 4/10 | 0.67032 | 13/10 | 0.27253 | 16/5 | 0.04076 | 9 | 0.00012 |
| 5/10 | 0.60653 | 4/3 | 0.26360 | 18/5 | 0.02732 | 10 | 0.00004 |
| 6/10 | 0.54881 | 14/10 | 0.24660 | 4 | 0.01832 | 11 | 0.00002 |
| 2/3 | 0.51342 | 3/2 | 0.22313 | 25/6 | 0.01550 | 12 | 0.00001 |
| 7/10 | 0.49659 | 16/10 | 0.20190 | 9/2 | 0.01111 | 13 | 0.00000 |

The Common Logarithms of e^x and e^{-x} .

| x | $\log_{10} e^x$ | $\log_{10} e^{-x}$ |
|---------|-----------------|--------------------|
| 0.00001 | 0.0000043429 | 1.9999956571 |
| 0.00002 | 0.0000086859 | 1.9999913141 |
| 0.00003 | 0.0000130288 | 1.9999869712 |
| 0.00004 | 0.0000173718 | 1.9999826282 |
| 0.00005 | 0.0000217147 | 1.9999782853 |
| 0.00006 | 0.0000260577 | 1.9999739423 |
| 0.00007 | 0.0000304006 | 1.9999695994 |
| 0.00008 | 0.0000347436 | 1.9999652564 |
| 0.00009 | 0.0000390865 | 1.9999609135 |
| 0.00010 | 0.0000434294 | 1.9999565706 |
| 0.00020 | 0.0000868589 | 1.9999131411 |
| 0.00030 | 0.0001302883 | 1.9998697117 |
| 0.00040 | 0.0001737178 | 1.9998262822 |
| 0.00050 | 0.0002171472 | 1.9997828528 |
| 0.00060 | 0.0002605767 | 1.9997394233 |
| 0.00070 | 0.0003040061 | 1.9996959939 |
| 0.00080 | 0.0003474356 | 1.9996525644 |
| 0.00090 | 0.0003908650 | 1.9996091350 |
| 0.00100 | 0.0004342945 | 1.9995657055 |
| 0.00200 | 0.0008685890 | 1.9991314110 |
| 0.00300 | 0.0013028834 | 1.9986971166 |
| 0.00400 | 0.0017371779 | 1.9982628221 |
| 0.00500 | 0.0021714724 | 1.9978285276 |
| 0.00600 | 0.0026057669 | 1.9973942331 |
| 0.00700 | 0.0030400614 | 1.9969599386 |
| 0.00800 | 0.0034743559 | 1.9965256441 |
| 0.00900 | 0.0039086503 | 1.9960913497 |
| 0.01000 | 0.0043429448 | 1.9956570552 |
| 0.02000 | 0.0086858896 | 1.9913141104 |
| 0.03000 | 0.0130288345 | 1.9869711655 |
| 0.04000 | 0.0173717793 | 1.9826282207 |
| 0.05000 | 0.0217147241 | 1.9782852759 |
| 0.06000 | 0.0260576689 | 1.9739423311 |
| 0.07000 | 0.0304006137 | 1.9695993863 |

| x | $\log_{10} e^x$ | $\log_{10} e^{-x}$ |
|-----------|-----------------|-----------------------|
| 0.08000 | 0.0347435586 | <u>1.9652564414</u> |
| 0.09000 | 0.0390865034 | <u>1.9609134966</u> |
| 0.10000 | 0.0434294482 | <u>1.9565705518</u> |
| 0.20000 | 0.0868588964 | <u>1.9131411036</u> |
| 0.30000 | 0.1302883446 | <u>1.8697116554</u> |
| 0.40000 | 0.1737177928 | <u>1.8262822072</u> |
| 0.50000 | 0.2171472410 | <u>1.7828527590</u> |
| 0.60000 | 0.2605766891 | <u>1.7394233109</u> |
| 0.70000 | 0.3040061373 | <u>1.6959938627</u> |
| 0.80000 | 0.3474355855 | <u>1.6525644145</u> |
| 0.90000 | 0.3908650337 | <u>1.6091349663</u> |
| 1.00000 | 0.4342944819 | <u>1.5657055181</u> |
| 2.00000 | 0.8685889638 | <u>1.1314110362</u> |
| 3.00000 | 1.3028834457 | <u>2.6971165543</u> |
| 4.00000 | 1.7371779276 | <u>2.2628220724</u> |
| 5.00000 | 2.1714724095 | <u>3.8285275905</u> |
| 6.00000 | 2.6057668914 | <u>3.3942331086</u> |
| 7.00000 | 3.0400613733 | <u>4.9599386267</u> |
| 8.00000 | 3.4743558552 | <u>4.5256441448</u> |
| 9.00000 | 3.9086503371 | <u>4.0913496629</u> |
| 10.00000 | 4.3429448190 | <u>5.6570551810</u> |
| 20.00000 | 8.6858896381 | <u>9.3141103619</u> |
| 30.00000 | 13.0288344571 | <u>14.9711655429</u> |
| 40.00000 | 17.3717792761 | <u>18.6282207239</u> |
| 50.00000 | 21.7147240952 | <u>22.2852759048</u> |
| 60.00000 | 26.0576689142 | <u>27.9423310858</u> |
| 70.00000 | 30.4006137332 | <u>31.5993862668</u> |
| 80.00000 | 34.7435585523 | <u>35.2564414477</u> |
| 90.00000 | 39.0865033713 | <u>40.9134966287</u> |
| 100.00000 | 43.4294481903 | <u>44.5705518097</u> |
| 200.00000 | 86.8588963807 | <u>87.1411036193</u> |
| 300.00000 | 130.2883445710 | <u>131.7116554290</u> |
| 400.00000 | 173.7177927613 | <u>174.2822072387</u> |
| 500.00000 | 217.1472409516 | <u>218.8527590484</u> |

Note: $\log e^{x+y} = \log e^x + \log e^y$. Thus, $\log e^{113.1478} = 49.139465180$.

Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|------|----------|------|-------|------|------|------|-------|------|------|------|--------|
| 1.00 | 0.0 0000 | 0100 | 0200 | 0300 | 0399 | 0499 | 0598 | 0698 | 0797 | 0896 | 100-99 |
| 1.01 | 0.0 0995 | 1094 | 1193 | 1292 | 1390 | 1489 | 1587 | 1686 | 1784 | 1882 | 99-98 |
| 1.02 | 0.0 1980 | 2078 | 2176 | 2274 | 2372 | 2469 | 2567 | 2664 | 2762 | 2859 | 98-97 |
| 1.03 | 0.0 2956 | 3053 | 3150 | 3247 | 3343 | 3440 | 3537 | 3633 | 3730 | 3826 | 97-96 |
| 1.04 | 0.0 3922 | 4018 | 4114 | 4210 | 4306 | 4402 | 4497 | 4593 | 4688 | 4784 | 96-95 |
| 1.05 | 0.0 4879 | 4974 | 5069 | 5164 | 5259 | 5354 | 5449 | 5543 | 5638 | 5733 | 95-94 |
| 1.06 | 0.0 5827 | 5921 | 6015 | 6110 | 6204 | 6297 | 6391 | 6485 | 6579 | 6672 | 94 |
| 1.07 | 0.0 6766 | 6859 | 6953 | 7046 | 7139 | 7232 | 7325 | 7418 | 7511 | 7603 | 93 |
| 1.08 | 0.0 7696 | 7789 | 7881 | 7973 | 8066 | 8158 | 8250 | 8342 | 8434 | 8526 | 93-92 |
| 1.09 | 0.0 8618 | 8709 | 8801 | 8893 | 8984 | 9075 | 9167 | 9258 | 9349 | 9430 | 92-91 |
| 1.10 | 0.0 9531 | 9622 | 9713 | 9803 | 9894 | 9985 | *0075 | 0165 | 0256 | 0346 | 91-90 |
| 1.11 | 0.1 0436 | 0526 | 0616 | 0706 | 0796 | 0885 | 0975 | 1065 | 1154 | 1244 | 90-89 |
| 1.12 | 0.1 1333 | 1422 | 1511 | 1600 | 1689 | 1778 | 1867 | 1956 | 2045 | 2133 | 89 |
| 1.13 | 0.1 2222 | 2310 | 2399 | 2487 | 2575 | 2663 | 2751 | 2839 | 2927 | 3015 | 88 |
| 1.14 | 0.1 3103 | 3191 | 3278 | 3366 | 3453 | 3540 | 3628 | 3715 | 3802 | 3889 | 88-87 |
| 1.15 | 0.1 3976 | 4063 | 4150 | 4237 | 4323 | 4410 | 4497 | 4583 | 4669 | 4756 | 87-86 |
| 1.16 | 0.1 4842 | 4928 | 5014 | 5100 | 5186 | 5272 | 5358 | 5444 | 5529 | 5615 | 86 |
| 1.17 | 0.1 5700 | 5786 | 5871 | 5956 | 6042 | 6127 | 6212 | 6297 | 6382 | 6467 | 85 |
| 1.18 | 0.1 6551 | 6636 | 6721 | 6805 | 6890 | 6974 | 7059 | 7143 | 7227 | 7311 | 85-84 |
| 1.19 | 0.1 7395 | 7479 | 7563 | 7647 | 7731 | 7815 | 7898 | 7982 | 8065 | 8149 | 84-83 |
| 1.20 | 0.1 8232 | 8315 | 8399 | 8482 | 8565 | 8648 | 8731 | 8814 | 8897 | 9979 | 83 |
| 1.21 | 0.1 9062 | 9145 | 9227 | 9310 | 9392 | 9474 | 9557 | 9639 | 9721 | 9803 | 83-82 |
| 1.22 | 0.1 9885 | 9967 | *0049 | 0131 | 0212 | 0294 | 0376 | 0457 | 0539 | 0620 | 82-81 |
| 1.23 | 0.2 0701 | 0783 | 0864 | 0945 | 1026 | 1107 | 1188 | 1269 | 1350 | 1430 | 81 |
| 1.24 | 0.2 1511 | 1592 | 1672 | 1753 | 1833 | 1914 | 1994 | 2074 | 2154 | 2234 | 81-80 |
| 1.25 | 0.2 2314 | 2394 | 2474 | 2554 | 2634 | 2714 | 2793 | 2873 | 2952 | 3032 | 80-79 |
| 1.26 | 0.2 3111 | 3191 | 3270 | 3349 | 3428 | 3507 | 3586 | 3665 | 3744 | 3823 | 79 |
| 1.27 | 0.2 3902 | 3980 | 4059 | 4138 | 4216 | 4295 | 4373 | 4451 | 4530 | 4608 | 79-78 |
| 1.28 | 0.2 4686 | 4764 | 4842 | 4920 | 4998 | 5076 | 5154 | 5231 | 5309 | 5387 | 78 |
| 1.29 | 0.2 5464 | 5542 | 5619 | 5697 | 5774 | 5811 | 5928 | 6005 | 6082 | 6159 | 77 |
| 1.30 | 0.2 6236 | 6313 | 6390 | 6467 | 6544 | 6620 | 6697 | 6773 | 6850 | 6926 | 77-76 |
| 1.31 | 0.2 7003 | 7079 | 7155 | 7231 | 7308 | 7384 | 7460 | 7536 | 7612 | 7687 | 76 |
| 1.32 | 0.2 7763 | 7839 | 7915 | 7990 | 8066 | 8141 | 8217 | 8292 | 8367 | 8443 | 76-75 |
| 1.33 | 0.2 8518 | 8593 | 8668 | 8743 | 8818 | 8893 | 8968 | 9043 | 9118 | 9192 | 75 |
| 1.34 | 0.2 9267 | 9342 | 9416 | 9491 | 9565 | 9639 | 9714 | 9788 | 9862 | 9936 | 75-74 |
| 1.35 | 0.3 0010 | 0085 | 0158 | 0232 | 0306 | 0380 | 0454 | 0528 | 0601 | 0675 | 74 |
| 1.36 | 0.3 0748 | 0822 | 0895 | 0969 | 1042 | 1115 | 1189 | 1262 | 1335 | 1408 | 74-73 |
| 1.37 | 0.3 1481 | 1554 | 1627 | 1700 | 1773 | 1845 | 1918 | 1991 | 2063 | 2136 | 73-72 |
| 1.38 | 0.3 2208 | 2281 | 2353 | 2426 | 2498 | 2570 | 2642 | 2714 | 2786 | 2858 | 72 |
| 1.39 | 0.3 2930 | 3002 | 3074 | 3146 | 3218 | 3289 | 3361 | 3433 | 3504 | 3576 | 72-71 |
| 1.40 | 0.3 3647 | 3719 | 3790 | 3861 | 3933 | 4004 | 4075 | 4146 | 4217 | 4288 | 71 |
| 1.41 | 0.3 4359 | 4430 | 4501 | 4572 | 4642 | 4713 | 4784 | 4854 | 4925 | 4995 | 71-70 |
| 1.42 | 0.3 5066 | 5136 | 5206 | 5277 | 5347 | 5417 | 5487 | 5557 | 5627 | 5697 | 70 |
| 1.43 | 0.3 5767 | 5837 | 5907 | 5977 | 6047 | 6116 | 6186 | 6256 | 6325 | 6395 | 70-69 |
| 1.44 | 0.3 6464 | 6534 | 6603 | 6672 | 6742 | 6811 | 6880 | 6949 | 7018 | 7087 | 69 |
| 1.45 | 0.3 7156 | 7225 | 7294 | 7363 | 7432 | 7501 | 7569 | 7638 | 7707 | 7775 | 69 |
| 1.46 | 0.3 7844 | 7912 | 7981 | 8049 | 8117 | 8186 | 8254 | 8322 | 8390 | 8458 | 68 |
| 1.47 | 0.3 8526 | 8594 | 8662 | 8730 | 8798 | 8866 | 8934 | 9001 | 9069 | 9137 | 68 |
| 1.48 | 0.3 9204 | 9272 | 9339 | 9407 | 9474 | 9541 | 9609 | 9676 | 9743 | 9810 | 68-67 |
| 1.49 | 0.3 9878 | 9945 | *0012 | 0079 | 0146 | 0213 | 0279 | 0346 | 0413 | 0480 | 67 |
| 1.50 | 0.4 0547 | 0613 | 0680 | 0746 | 0813 | 0879 | 0946 | 1012 | 1078 | 1145 | 67-66 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|------|----------|------|------|-------|------|------|------|------|------|-------|-------|
| 1.50 | 0.4 0547 | 0613 | 0680 | 0746 | 0813 | 0879 | 0946 | 1012 | 1078 | 1145 | 67-66 |
| 1.51 | 0.4 1211 | 1277 | 1343 | 1409 | 1476 | 1542 | 1608 | 1673 | 1739 | 1805 | 66 |
| 1.52 | 0.4 1871 | 1937 | 2003 | 2068 | 2134 | 2199 | 2265 | 2331 | 2396 | 2461 | 66-65 |
| 1.53 | 0.4 2527 | 2592 | 2657 | 2723 | 2788 | 2853 | 2918 | 2983 | 3048 | 3113 | 65 |
| 1.54 | 0.4 3178 | 3243 | 3308 | 3373 | 3438 | 3502 | 3567 | 3632 | 3696 | 3761 | 65-64 |
| 1.55 | 0.4 3825 | 3890 | 3954 | 4019 | 4083 | 4148 | 4212 | 4276 | 4340 | 4404 | 64 |
| 1.56 | 0.4 4469 | 4533 | 4597 | 4661 | 4725 | 4789 | 4852 | 4916 | 4980 | 5044 | 64 |
| 1.57 | 0.4 5108 | 5171 | 5235 | 5298 | 5362 | 5426 | 5489 | 5552 | 5616 | 5679 | 64-63 |
| 1.58 | 0.4 5742 | 5806 | 5869 | 5932 | 5995 | 6058 | 6122 | 6185 | 6248 | 6310 | 63 |
| 1.59 | 0.4 6373 | 6436 | 6499 | 6562 | 6625 | 6687 | 6750 | 6813 | 6875 | 6938 | 63 |
| 1.60 | 0.4 7000 | 7063 | 7125 | 7188 | 7250 | 7312 | 7375 | 7437 | 7499 | 7561 | 62 |
| 1.61 | 0.4 7623 | 7686 | 7748 | 7810 | 7872 | 7933 | 7995 | 8057 | 8119 | 8181 | 62 |
| 1.62 | 0.4 8243 | 8304 | 8366 | 8428 | 8489 | 8551 | 8612 | 8674 | 8735 | 8797 | 62-61 |
| 1.63 | 0.4 8858 | 8919 | 8981 | 9042 | 9103 | 9164 | 9225 | 9287 | 9348 | 9409 | 61 |
| 1.64 | 0.4 9470 | 9531 | 9592 | 9652 | 9713 | 9774 | 9835 | 9896 | 9956 | *0017 | 61 |
| 1.65 | 0.5 0078 | 0138 | 0199 | 0259 | 0320 | 0380 | 0441 | 0501 | 0561 | 0622 | 61-60 |
| 1.66 | 0.5 0682 | 0742 | 0802 | 0862 | 0922 | 0983 | 1043 | 1103 | 1163 | 1222 | 60 |
| 1.67 | 0.5 1282 | 1342 | 1402 | 1462 | 1522 | 1581 | 1641 | 1701 | 1760 | 1820 | 60 |
| 1.68 | 0.5 1879 | 1939 | 1998 | 2058 | 2117 | 2177 | 2236 | 2295 | 2354 | 2414 | 60-59 |
| 1.69 | 0.5 2473 | 2532 | 2591 | 2650 | 2709 | 2768 | 2827 | 2886 | 2945 | 3004 | 59 |
| 1.70 | 0.5 3063 | 3122 | 3180 | 3239 | 3298 | 3357 | 3415 | 3474 | 3532 | 3591 | 59-58 |
| 1.71 | 0.5 3649 | 3708 | 3766 | 3825 | 3883 | 3941 | 4000 | 4058 | 4116 | 4174 | 58 |
| 1.72 | 0.5 4232 | 4291 | 4349 | 4407 | 4465 | 4523 | 4581 | 4639 | 4696 | 4754 | 58 |
| 1.73 | 0.5 4812 | 4870 | 4928 | 4985 | 5043 | 5101 | 5158 | 5216 | 5274 | 5331 | 58-57 |
| 1.74 | 0.5 5389 | 5446 | 5503 | 5561 | 5618 | 5675 | 5733 | 5790 | 5847 | 5904 | 57 |
| 1.75 | 0.5 5962 | 6019 | 6076 | 6133 | 6190 | 6247 | 6304 | 6361 | 6418 | 6475 | 57 |
| 1.76 | 0.5 6531 | 6588 | 6645 | 6702 | 6758 | 6815 | 6872 | 6928 | 6985 | 7041 | 57 |
| 1.77 | 0.5 7098 | 7154 | 7211 | 7267 | 7324 | 7380 | 7436 | 7493 | 7549 | 7605 | 56 |
| 1.78 | 0.5 7661 | 7718 | 7774 | 7830 | 7886 | 7942 | 7998 | 8054 | 8110 | 8166 | 56 |
| 1.79 | 0.5 8222 | 8277 | 8333 | 8389 | 8445 | 8501 | 8556 | 8612 | 8667 | 8723 | 56 |
| 1.80 | 0.5 8779 | 8834 | 8890 | 8945 | 9001 | 9056 | 9111 | 9167 | 9222 | 9277 | 56-55 |
| 1.81 | 0.5 9333 | 9388 | 9443 | 9498 | 9553 | 9609 | 9664 | 9719 | 9774 | 9829 | 55 |
| 1.82 | 0.5 9884 | 9939 | 9993 | *0048 | 0103 | 0158 | 0213 | 0268 | 0322 | 0377 | 55 |
| 1.83 | 0.6 0432 | 0486 | 0541 | 0595 | 0650 | 0704 | 0759 | 0813 | 0868 | 0922 | 55-54 |
| 1.84 | 0.6 0977 | 1031 | 1085 | 1139 | 1194 | 1248 | 1302 | 1356 | 1410 | 1464 | 54 |
| 1.85 | 0.6 1519 | 1573 | 1627 | 1681 | 1735 | 1788 | 1842 | 1896 | 1950 | 2004 | 54 |
| 1.86 | 0.6 2058 | 2111 | 2165 | 2219 | 2272 | 2326 | 2380 | 2433 | 2487 | 2540 | 54-53 |
| 1.87 | 0.6 2594 | 2647 | 2701 | 2754 | 2808 | 2861 | 2914 | 2967 | 3021 | 3074 | 53 |
| 1.88 | 0.6 3127 | 3180 | 3234 | 3287 | 3340 | 3393 | 3446 | 3499 | 3552 | 3605 | 53 |
| 1.89 | 0.6 3658 | 3711 | 3763 | 3816 | 3869 | 3922 | 3975 | 4027 | 4080 | 4133 | 53 |
| 1.90 | 0.6 4185 | 4238 | 4291 | 4343 | 4396 | 4448 | 4501 | 4553 | 4606 | 4658 | 53-52 |
| 1.91 | 0.6 4710 | 4763 | 4815 | 4867 | 4920 | 4972 | 5024 | 5076 | 5128 | 5180 | 52 |
| 1.92 | 0.6 5233 | 5285 | 5337 | 5389 | 5441 | 5493 | 5545 | 5596 | 5648 | 5700 | 52 |
| 1.93 | 0.6 5752 | 5804 | 5856 | 5907 | 5959 | 6011 | 6062 | 6114 | 6166 | 6217 | 52 |
| 1.94 | 0.6 6269 | 6320 | 6372 | 6423 | 6475 | 6526 | 6578 | 6629 | 6680 | 6732 | 52-51 |
| 1.95 | 0.6 6783 | 6834 | 6885 | 6937 | 6988 | 7039 | 7090 | 7141 | 7192 | 7243 | 51 |
| 1.96 | 0.6 7294 | 7345 | 7396 | 7447 | 7498 | 7549 | 7600 | 7651 | 7702 | 7753 | 51 |
| 1.97 | 0.6 7803 | 7854 | 7905 | 7956 | 8006 | 8057 | 8107 | 8158 | 8209 | 8259 | 51 |
| 1.98 | 0.6 8310 | 8360 | 8411 | 8461 | 8512 | 8562 | 8612 | 8663 | 8713 | 8763 | 50 |
| 1.99 | 0.6 8813 | 8864 | 8914 | 8964 | 9014 | 9064 | 9115 | 9165 | 9215 | 9265 | 50 |
| 2.00 | 0.6 9315 | 9365 | 9415 | 9465 | 9515 | 9564 | 9614 | 9664 | 9714 | 9764 | 50 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|------|----------|------|------|------|-------|------|-------|------|------|------|-------|
| 2.00 | 0.6 9315 | 9365 | 9415 | 9465 | 9515 | 9564 | 9614 | 9664 | 9714 | 9764 | 50 |
| 2.01 | 0.6 9813 | 9863 | 9913 | 9963 | *0012 | 0062 | 0112 | 0161 | 0211 | 0260 | 50 |
| 2.02 | 0.7 0310 | 0359 | 0409 | 0458 | 0508 | 0557 | 0606 | 0656 | 0705 | 0754 | 49 |
| 2.03 | 0.7 0804 | 0853 | 0902 | 0951 | 1000 | 1050 | 1099 | 1148 | 1197 | 1246 | 49 |
| 2.04 | 0.7 1295 | 1344 | 1393 | 1442 | 1491 | 1540 | 1589 | 1638 | 1686 | 1735 | 49 |
| 2.05 | 0.7 1784 | 1833 | 1881 | 1930 | 1979 | 2028 | 2076 | 2125 | 2173 | 2222 | 49 |
| 2.06 | 0.7 2271 | 2319 | 2368 | 2416 | 2465 | 2513 | 2561 | 2610 | 2658 | 2707 | 49-48 |
| 2.07 | 0.7 2755 | 2803 | 2851 | 2900 | 2948 | 2996 | 3044 | 3092 | 3141 | 3189 | 48 |
| 2.08 | 0.7 3237 | 3285 | 3333 | 3381 | 3429 | 3477 | 3525 | 3573 | 3621 | 3669 | 48 |
| 2.09 | 0.7 3716 | 3764 | 3812 | 3860 | 3908 | 3955 | 4003 | 4051 | 4098 | 4146 | 48 |
| 2.10 | 0.7 4194 | 4241 | 4289 | 4336 | 4384 | 4432 | 4479 | 4527 | 4574 | 4621 | 48-47 |
| 2.11 | 0.7 4669 | 4716 | 4764 | 4811 | 4858 | 4905 | 4953 | 5000 | 5047 | 5094 | 47 |
| 2.12 | 0.7 5142 | 5189 | 5236 | 5283 | 5330 | 5377 | 5424 | 5471 | 5518 | 5565 | 47 |
| 2.13 | 0.7 5612 | 5659 | 5706 | 5753 | 5800 | 5847 | 5893 | 5940 | 5987 | 6034 | 47 |
| 2.14 | 0.7 6081 | 6127 | 6174 | 6221 | 6267 | 6314 | 6361 | 6407 | 6454 | 6500 | 47 |
| 2.15 | 0.7 6547 | 6593 | 6640 | 6686 | 6733 | 6779 | 6825 | 6872 | 6918 | 6965 | 47-46 |
| 2.16 | 0.7 7011 | 7057 | 7103 | 7150 | 7196 | 7242 | 7288 | 7334 | 7381 | 7427 | 46 |
| 2.17 | 0.7 7473 | 7519 | 7565 | 7611 | 7657 | 7703 | 7749 | 7795 | 7841 | 7887 | 46 |
| 2.18 | 0.7 7932 | 7978 | 8024 | 8070 | 8116 | 8162 | 8207 | 8253 | 8299 | 8344 | 46 |
| 2.19 | 0.7 8390 | 8436 | 8481 | 8527 | 8573 | 8618 | 8664 | 8709 | 8755 | 8800 | 46-45 |
| 2.20 | 0.7 8846 | 8891 | 8937 | 8982 | 9027 | 9073 | 9118 | 9163 | 9209 | 9254 | 45 |
| 2.21 | 0.7 9299 | 9344 | 9390 | 9435 | 9480 | 9525 | 9570 | 9615 | 9661 | 9706 | 45 |
| 2.22 | 0.7 9751 | 9796 | 9841 | 9886 | 9931 | 9976 | *0021 | 0066 | 0110 | 0155 | 45 |
| 2.23 | 0.8 0200 | 0245 | 0290 | 0335 | 0379 | 0424 | 0469 | 0514 | 0558 | 0603 | 45 |
| 2.24 | 0.8 0648 | 0692 | 0737 | 0781 | 0826 | 0871 | 0915 | 0960 | 1004 | 1049 | 45-44 |
| 2.25 | 0.8 1093 | 1137 | 1182 | 1226 | 1271 | 1315 | 1359 | 1404 | 1448 | 1492 | 44 |
| 2.26 | 0.8 1536 | 1581 | 1625 | 1669 | 1713 | 1757 | 1802 | 1846 | 1890 | 1934 | 44 |
| 2.27 | 0.8 1978 | 2022 | 2066 | 2110 | 2154 | 2198 | 2242 | 2286 | 2330 | 2374 | 44 |
| 2.28 | 0.8 2418 | 2461 | 2505 | 2549 | 2593 | 2637 | 2680 | 2724 | 2768 | 2812 | 44 |
| 2.29 | 0.8 2855 | 2899 | 2942 | 2986 | 3030 | 3073 | 3117 | 3160 | 3204 | 3247 | 44-43 |
| 2.30 | 0.8 3291 | 3334 | 3378 | 3421 | 3465 | 3508 | 3551 | 3595 | 3638 | 3681 | 43 |
| 2.31 | 0.8 3725 | 3768 | 3811 | 3855 | 3898 | 3941 | 3984 | 4027 | 4070 | 4114 | 43 |
| 2.32 | 0.8 4157 | 4200 | 4243 | 4286 | 4329 | 4372 | 4415 | 4458 | 4501 | 4544 | 43 |
| 2.33 | 0.8 4587 | 4630 | 4673 | 4715 | 4758 | 4801 | 4844 | 4887 | 4930 | 4972 | 43 |
| 2.34 | 0.8 5015 | 5058 | 5101 | 5143 | 5186 | 5229 | 5271 | 5314 | 5356 | 5399 | 43 |
| 2.35 | 0.8 5442 | 5484 | 5527 | 5569 | 5612 | 5654 | 5697 | 5739 | 5781 | 5824 | 43-42 |
| 2.36 | 0.8 5866 | 5909 | 5951 | 5993 | 6036 | 6078 | 6120 | 6162 | 6205 | 6247 | 42 |
| 2.37 | 0.8 6289 | 6331 | 6373 | 6415 | 6458 | 6500 | 6542 | 6584 | 6626 | 6668 | 42 |
| 2.38 | 0.8 6710 | 6752 | 6794 | 6836 | 6878 | 6920 | 6962 | 7004 | 7046 | 7087 | 42 |
| 2.39 | 0.8 7129 | 7171 | 7213 | 7255 | 7297 | 7338 | 7380 | 7422 | 7464 | 7505 | 42 |
| 2.40 | 0.8 7547 | 7589 | 7630 | 7672 | 7713 | 7755 | 7797 | 7838 | 7880 | 7921 | 42 |
| 2.41 | 0.8 7963 | 8004 | 8046 | 8087 | 8129 | 8170 | 8211 | 8253 | 8294 | 8335 | 41 |
| 2.42 | 0.8 8377 | 8418 | 8459 | 8501 | 8542 | 8583 | 8624 | 8666 | 8707 | 8748 | 41 |
| 2.43 | 0.8 8789 | 8830 | 8871 | 8913 | 8954 | 8995 | 9036 | 9077 | 9118 | 9159 | 41 |
| 2.44 | 0.8 9200 | 9241 | 9282 | 9323 | 9364 | 9405 | 9445 | 9486 | 9527 | 9568 | 41 |
| 2.45 | 0.8 9609 | 9650 | 9690 | 9731 | 9772 | 9813 | 9853 | 9894 | 9935 | 9975 | 41 |
| 2.46 | 0.9 0016 | 0057 | 0097 | 0138 | 0179 | 0219 | 0260 | 0300 | 0341 | 0381 | 41-40 |
| 2.47 | 0.9 0422 | 0462 | 0503 | 0543 | 0584 | 0624 | 0664 | 0705 | 0745 | 0786 | 40 |
| 2.48 | 0.9 0826 | 0866 | 0906 | 0947 | 0987 | 1027 | 1067 | 1108 | 1148 | 1188 | 40 |
| 2.49 | 0.9 1228 | 1268 | 1309 | 1349 | 1389 | 1429 | 1469 | 1509 | 1549 | 1589 | 40 |
| 2.50 | 0.9 1629 | 1669 | 1709 | 1749 | 1789 | 1829 | 1869 | 1909 | 1949 | 1988 | 40 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|------|----------|------|------|------|------|-------|------|------|------|-------|-------|
| 2.50 | 0.9 1629 | 1669 | 1709 | 1749 | 1789 | 1829 | 1869 | 1909 | 1949 | 1988 | 40 |
| 2.51 | 0.9 2028 | 2068 | 2108 | 2148 | 2188 | 2227 | 2267 | 2307 | 2346 | 2386 | 40 |
| 2.52 | 0.9 2426 | 2466 | 2505 | 2545 | 2584 | 2624 | 2664 | 2703 | 2743 | 2782 | 40 |
| 2.53 | 0.9 2822 | 2861 | 2901 | 2940 | 2980 | 3019 | 3059 | 3098 | 3138 | 3177 | 40-39 |
| 2.54 | 0.9 3216 | 3256 | 3295 | 3334 | 3374 | 3413 | 3452 | 3492 | 3531 | 3570 | 39 |
| 2.55 | 0.9 3609 | 3649 | 3688 | 3727 | 3766 | 3805 | 3844 | 3883 | 3923 | 3962 | 39 |
| 2.56 | 0.9 4001 | 4040 | 4079 | 4118 | 4157 | 4196 | 4235 | 4274 | 4313 | 4352 | 39 |
| 2.57 | 0.9 4391 | 4429 | 4468 | 4507 | 4546 | 4585 | 4624 | 4663 | 4701 | 4740 | 39 |
| 2.58 | 0.9 4779 | 4818 | 4856 | 4895 | 4934 | 4973 | 5011 | 5050 | 5089 | 5127 | 39 |
| 2.59 | 0.9 5166 | 5204 | 5243 | 5282 | 5320 | 5359 | 5397 | 5436 | 5474 | 5513 | 39-38 |
| 2.60 | 0.9 5551 | 5590 | 5628 | 5666 | 5705 | 5743 | 5782 | 5820 | 5858 | 5897 | 38 |
| 2.61 | 0.9 5935 | 5973 | 6012 | 6050 | 6088 | 6126 | 6165 | 6203 | 6241 | 6279 | 38 |
| 2.62 | 0.9 6317 | 6356 | 6394 | 6432 | 6470 | 6508 | 6546 | 6584 | 6622 | 6660 | 38 |
| 2.63 | 0.9 6698 | 6736 | 6774 | 6812 | 6850 | 6888 | 6926 | 6964 | 7002 | 7040 | 38 |
| 2.64 | 0.9 7078 | 7116 | 7154 | 7191 | 7229 | 7267 | 7305 | 7343 | 7380 | 7418 | 38 |
| 2.65 | 0.9 7456 | 7494 | 7531 | 7569 | 7607 | 7644 | 7682 | 7720 | 7757 | 7795 | 38 |
| 2.66 | 0.9 7833 | 7870 | 7908 | 7945 | 7983 | 8020 | 8058 | 8095 | 8133 | 8170 | 38-37 |
| 2.67 | 0.9 8208 | 8245 | 8283 | 8320 | 8358 | 8395 | 8432 | 8470 | 8507 | 8544 | 37 |
| 2.68 | 0.9 8582 | 8619 | 8656 | 8694 | 8731 | 8768 | 8805 | 8843 | 8880 | 8917 | 37 |
| 2.69 | 0.9 8954 | 8991 | 9028 | 9066 | 9103 | 9140 | 9177 | 9214 | 9251 | 9288 | 37 |
| 2.70 | 0.9 9325 | 9362 | 9399 | 9436 | 9473 | 9510 | 9547 | 9584 | 9621 | 9658 | 37 |
| 2.71 | 0.9 9695 | 9732 | 9769 | 9806 | 9842 | 9879 | 9916 | 9953 | 9990 | *0026 | 37 |
| 2.72 | 1.0 0063 | 0100 | 0137 | 0173 | 0210 | 0247 | 0284 | 0320 | 0357 | 0394 | 37 |
| 2.73 | 1.0 0430 | 0467 | 0503 | 0540 | 0577 | 0613 | 0650 | 0686 | 0723 | 0759 | 37 |
| 2.74 | 1.0 0796 | 0832 | 0869 | 0905 | 0942 | 0978 | 1015 | 1051 | 1087 | 1124 | 36 |
| 2.75 | 1.0 1160 | 1196 | 1233 | 1269 | 1305 | 1342 | 1378 | 1414 | 1451 | 1487 | 36 |
| 2.76 | 1.0 1523 | 1559 | 1596 | 1632 | 1668 | 1704 | 1740 | 1776 | 1813 | 1849 | 36 |
| 2.77 | 1.0 1885 | 1921 | 1957 | 1993 | 2029 | 2065 | 2101 | 2137 | 2173 | 2209 | 36 |
| 2.78 | 1.0 2245 | 2281 | 2317 | 2353 | 2389 | 2425 | 2461 | 2497 | 2532 | 2568 | 36 |
| 2.79 | 1.0 2604 | 2640 | 2676 | 2712 | 2747 | 2783 | 2819 | 2855 | 2890 | 2926 | 36 |
| 2.80 | 1.0 2962 | 2998 | 3033 | 3069 | 3105 | 3140 | 3176 | 3212 | 3247 | 3283 | 36 |
| 2.81 | 1.0 3318 | 3354 | 3390 | 3425 | 3461 | 3496 | 3532 | 3567 | 3603 | 3638 | 36-35 |
| 2.82 | 1.0 3674 | 3709 | 3745 | 3780 | 3815 | 3851 | 3886 | 3922 | 3957 | 3992 | 35 |
| 2.83 | 1.0 4028 | 4063 | 4098 | 4134 | 4169 | 4204 | 4239 | 4275 | 4310 | 4345 | 35 |
| 2.84 | 1.0 4380 | 4416 | 4451 | 4486 | 4521 | 4556 | 4591 | 4627 | 4662 | 4697 | 35 |
| 2.85 | 1.0 4732 | 4767 | 4802 | 4837 | 4872 | 4907 | 4942 | 4977 | 5012 | 5047 | 35 |
| 2.86 | 1.0 5082 | 5117 | 5152 | 5187 | 5222 | 5257 | 5292 | 5327 | 5361 | 5396 | 35 |
| 2.87 | 1.0 5431 | 5466 | 5501 | 5536 | 5570 | 5605 | 5640 | 5675 | 5710 | 5744 | 35 |
| 2.88 | 1.0 5779 | 5814 | 5848 | 5883 | 5918 | 5952 | 5987 | 6022 | 6056 | 6091 | 35 |
| 2.89 | 1.0 6126 | 6160 | 6195 | 6229 | 6264 | 6299 | 6333 | 6368 | 6402 | 6437 | 35-34 |
| 2.90 | 1.0 6471 | 6506 | 6540 | 6574 | 6609 | 6643 | 6678 | 6712 | 6747 | 6781 | 34 |
| 2.91 | 1.0 6815 | 6850 | 6884 | 6918 | 6953 | 6987 | 7021 | 7056 | 7090 | 7124 | 34 |
| 2.92 | 1.0 7158 | 7193 | 7227 | 7261 | 7295 | 7329 | 7364 | 7398 | 7432 | 7466 | 34 |
| 2.93 | 1.0 7500 | 7534 | 7568 | 7603 | 7637 | 7671 | 7705 | 7739 | 7773 | 7807 | 34 |
| 2.94 | 1.0 7841 | 7875 | 7909 | 7943 | 7977 | 8011 | 8045 | 8079 | 8113 | 8147 | 34 |
| 2.95 | 1.0 8181 | 8214 | 8248 | 8282 | 8316 | 8350 | 8384 | 8418 | 8451 | 8485 | 34 |
| 2.96 | 1.0 8519 | 8553 | 8586 | 8620 | 8654 | 8688 | 8721 | 8755 | 8789 | 8823 | 34 |
| 2.97 | 1.0 8856 | 8890 | 8924 | 8957 | 8991 | 9024 | 9058 | 9092 | 9125 | 9159 | 34 |
| 2.98 | 1.0 9192 | 9226 | 9259 | 9293 | 9326 | 9360 | 9393 | 9427 | 9460 | 9494 | 34-33 |
| 2.99 | 1.0 9527 | 9561 | 9594 | 9628 | 9661 | 9694 | 9728 | 9761 | 9795 | 9828 | 33 |
| 3.00 | 1.0 9861 | 9895 | 9928 | 9961 | 9994 | *0028 | 0061 | 0094 | 0128 | 0161 | 33 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|------|----------|-------|------|------|------|-------|------|------|------|------|-------|
| 3.00 | 1.0 9861 | 9895 | 9928 | 9961 | 9994 | *0028 | 0061 | 0094 | 0128 | 0161 | 33 |
| 3.01 | 1.1 0194 | 0227 | 0260 | 0294 | 0327 | 0360 | 0393 | 0426 | 0459 | 0493 | 33 |
| 3.02 | 1.1 0526 | 0559 | 0592 | 0625 | 0658 | 0691 | 0724 | 0757 | 0790 | 0823 | 33 |
| 3.03 | 1.1 0856 | 0889 | 0922 | 0955 | 0988 | 1021 | 1054 | 1087 | 1120 | 1153 | 33 |
| 3.04 | 1.1 1186 | 1219 | 1252 | 1284 | 1317 | 1350 | 1383 | 1416 | 1449 | 1481 | 33 |
| 3.05 | 1.1 1514 | 1547 | 1580 | 1612 | 1645 | 1678 | 1711 | 1743 | 1776 | 1809 | 33 |
| 3.06 | 1.1 1841 | 1874 | 1907 | 1939 | 1972 | 2005 | 2037 | 2070 | 2103 | 2135 | 33 |
| 3.07 | 1.1 2168 | 2200 | 2233 | 2265 | 2298 | 2330 | 2363 | 2396 | 2428 | 2460 | 33-32 |
| 3.08 | 1.1 2493 | 2525 | 2558 | 2590 | 2623 | 2655 | 2688 | 2720 | 2752 | 2785 | 32 |
| 3.09 | 1.1 2817 | 2849 | 2882 | 2914 | 2946 | 2979 | 3011 | 3043 | 3076 | 3108 | 32 |
| 3.10 | 1.1 3140 | 3172 | 3205 | 3237 | 3269 | 3301 | 3334 | 3366 | 3398 | 3430 | 32 |
| 3.11 | 1.1 3462 | 3494 | 3527 | 3559 | 3591 | 3623 | 3655 | 3687 | 3719 | 3751 | 32 |
| 3.12 | 1.1 3783 | 3815 | 3847 | 3879 | 3911 | 3943 | 3955 | 4007 | 4039 | 4071 | 32 |
| 3.13 | 1.1 4103 | 4135 | 4167 | 4199 | 4231 | 4263 | 4295 | 4327 | 4359 | 4390 | 32 |
| 3.14 | 1.1 4422 | 4454 | 4486 | 4518 | 4550 | 4581 | 4613 | 4645 | 4677 | 4708 | 32 |
| 3.15 | 1.1 4740 | 4772 | 4804 | 4835 | 4867 | 4899 | 4931 | 4962 | 4994 | 5026 | 32 |
| 3.16 | 1.1 5057 | 5089 | 5120 | 5152 | 5184 | 5215 | 5247 | 5278 | 5310 | 5342 | 32 |
| 3.17 | 1.1 5373 | 5405 | 5436 | 5468 | 5499 | 5531 | 5562 | 5594 | 5625 | 5657 | 32-31 |
| 3.18 | 1.1 5688 | 5720 | 5751 | 5782 | 5814 | 5845 | 5877 | 5908 | 5939 | 5971 | 31 |
| 3.19 | 1.1 6002 | 6033 | 6065 | 6096 | 6127 | 6159 | 6190 | 6221 | 6253 | 6284 | 31 |
| 3.20 | 1.1 6315 | 6346 | 6378 | 6409 | 6440 | 6471 | 6502 | 6534 | 6565 | 6596 | 31 |
| 3.21 | 1.1 6627 | 6658 | 6689 | 6721 | 6752 | 6783 | 6814 | 6845 | 6876 | 6907 | 31 |
| 3.22 | 1.1 6938 | 6969 | 7000 | 7031 | 7062 | 7093 | 7124 | 7155 | 7186 | 7217 | 31 |
| 3.23 | 1.1 7248 | 7279 | 7310 | 7341 | 7372 | 7403 | 7434 | 7465 | 7496 | 7526 | 31 |
| 3.24 | 1.1 7557 | 7588 | 7619 | 7650 | 7681 | 7712 | 7742 | 7773 | 7804 | 7835 | 31 |
| 3.25 | 1.1 7865 | 7896 | 7927 | 7958 | 7989 | 8019 | 8050 | 8081 | 8111 | 8142 | 31 |
| 3.26 | 1.1 8173 | 8203 | 8234 | 8265 | 8295 | 8326 | 8357 | 8387 | 8418 | 8448 | 31 |
| 3.27 | 1.1 8479 | 8510 | 8540 | 8571 | 8601 | 8632 | 8662 | 8693 | 8723 | 8754 | 31-30 |
| 3.28 | 1.1 8784 | 8815 | 8845 | 8876 | 8906 | 8937 | 8967 | 8998 | 9028 | 9058 | 30 |
| 3.29 | 1.1 9089 | 9119 | 9150 | 9180 | 9210 | 9241 | 9271 | 9301 | 9332 | 9362 | 30 |
| 3.30 | 1.1 9392 | 9423 | 9453 | 9483 | 9513 | 9544 | 9574 | 9604 | 9634 | 9665 | 30 |
| 3.31 | 1.1 9695 | 9725 | 9755 | 9785 | 9816 | 9846 | 9876 | 9906 | 9936 | 9966 | 30 |
| 3.32 | 1.1 9996 | *0027 | 0057 | 0087 | 0117 | 0147 | 0177 | 0207 | 0237 | 0267 | 30 |
| 3.33 | 1.2 0297 | 0327 | 0357 | 0387 | 0417 | 0447 | 0477 | 0507 | 0537 | 0567 | 30 |
| 3.34 | 1.2 0597 | 0627 | 0657 | 0687 | 0717 | 0747 | 0777 | 0806 | 0836 | 0866 | 30 |
| 3.35 | 1.2 0896 | 0926 | 0956 | 0986 | 1015 | 1045 | 1075 | 1105 | 1135 | 1164 | 30 |
| 3.36 | 1.2 1194 | 1224 | 1254 | 1283 | 1313 | 1343 | 1373 | 1402 | 1432 | 1462 | 30 |
| 3.37 | 1.2 1491 | 1521 | 1551 | 1580 | 1610 | 1640 | 1669 | 1699 | 1728 | 1758 | 30 |
| 3.38 | 1.2 1788 | 1817 | 1847 | 1876 | 1906 | 1935 | 1965 | 1994 | 2024 | 2053 | 30 |
| 3.39 | 1.2 2083 | 2112 | 2142 | 2171 | 2201 | 2230 | 2260 | 2289 | 2319 | 2348 | 29 |
| 3.40 | 1.2 2378 | 2407 | 2436 | 2466 | 2495 | 2524 | 2554 | 2583 | 2613 | 2642 | 29 |
| 3.41 | 1.2 2671 | 2701 | 2730 | 2759 | 2788 | 2818 | 2847 | 2876 | 2906 | 2935 | 29 |
| 3.42 | 1.2 2964 | 2993 | 3023 | 3052 | 3081 | 3110 | 3139 | 3169 | 3198 | 3227 | 29 |
| 3.43 | 1.2 3256 | 3285 | 3314 | 3343 | 3373 | 3402 | 3431 | 3460 | 3489 | 3518 | 29 |
| 3.44 | 1.2 3547 | 3576 | 3605 | 3634 | 3663 | 3692 | 3721 | 3750 | 3779 | 3808 | 29 |
| 3.45 | 1.2 3837 | 3866 | 3895 | 3924 | 3953 | 3982 | 4011 | 4040 | 4069 | 4098 | 29 |
| 3.46 | 1.2 4127 | 4156 | 4185 | 4214 | 4242 | 4271 | 4300 | 4329 | 4358 | 4387 | 29 |
| 3.47 | 1.2 4415 | 4444 | 4473 | 4502 | 4531 | 4559 | 4588 | 4617 | 4646 | 4674 | 29 |
| 3.48 | 1.2 4703 | 4732 | 4761 | 4789 | 4818 | 4847 | 4875 | 4904 | 4933 | 4962 | 29 |
| 3.49 | 1.2 4990 | 5019 | 5047 | 5076 | 5105 | 5133 | 5162 | 5191 | 5219 | 5248 | 29 |
| 3.50 | 1.2 5276 | 5305 | 5333 | 5362 | 5391 | 5419 | 5448 | 5476 | 5505 | 5533 | 29-28 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|------|----------|------|------|------|------|------|------|------|------|------|-------|
| 3.50 | 1.2 5276 | 5305 | 5333 | 5362 | 5391 | 5419 | 5448 | 5476 | 5505 | 5533 | 29-28 |
| 3.51 | 1.2 5562 | 5590 | 5619 | 5647 | 5675 | 5704 | 5732 | 5761 | 5789 | 5818 | 28 |
| 3.52 | 1.2 5846 | 5875 | 5903 | 5931 | 5960 | 5988 | 6016 | 6045 | 6073 | 6101 | 28 |
| 3.53 | 1.2 6130 | 6158 | 6186 | 6215 | 6243 | 6271 | 6300 | 6328 | 6356 | 6384 | 28 |
| 3.54 | 1.2 6413 | 6441 | 6469 | 6497 | 6526 | 6554 | 6582 | 6610 | 6638 | 6667 | 28 |
| 3.55 | 1.2 6695 | 6723 | 6751 | 6779 | 6807 | 6836 | 6864 | 6892 | 6920 | 6948 | 28 |
| 3.56 | 1.2 6976 | 7004 | 7032 | 7060 | 7088 | 7116 | 7144 | 7172 | 7201 | 7229 | 28 |
| 3.57 | 1.2 7257 | 7285 | 7313 | 7341 | 7369 | 7397 | 7424 | 7452 | 7480 | 7508 | 28 |
| 3.58 | 1.2 7536 | 7564 | 7592 | 7620 | 7648 | 7676 | 7704 | 7732 | 7759 | 7787 | 28 |
| 3.59 | 1.2 7815 | 7843 | 7871 | 7899 | 7927 | 7954 | 7982 | 8010 | 8038 | 8066 | 28 |
| 3.60 | 1.2 8093 | 8121 | 8149 | 8177 | 8204 | 8232 | 8260 | 8288 | 8315 | 8343 | 28 |
| 3.61 | 1.2 8371 | 8398 | 8426 | 8454 | 8482 | 8509 | 8537 | 8564 | 8592 | 8620 | 28 |
| 3.62 | 1.2 8647 | 8675 | 8703 | 8730 | 8758 | 8785 | 8813 | 8841 | 8868 | 8896 | 28 |
| 3.63 | 1.2 8923 | 8951 | 8978 | 9006 | 9033 | 9061 | 9088 | 9116 | 9143 | 9171 | 28-27 |
| 3.64 | 1.2 9198 | 9226 | 9253 | 9281 | 9308 | 9336 | 9363 | 9390 | 9418 | 9445 | 27 |
| 3.65 | 1.2 9473 | 9500 | 9527 | 9555 | 9582 | 9610 | 9637 | 9664 | 9692 | 9719 | 27 |
| 3.66 | 1.2 9746 | 9774 | 9801 | 9828 | 9856 | 9883 | 9910 | 9937 | 9965 | 9992 | 27 |
| 3.67 | 1.3 0019 | 0046 | 0074 | 0101 | 0128 | 0155 | 0183 | 0210 | 0237 | 0264 | 27 |
| 3.68 | 1.3 0291 | 0318 | 0346 | 0373 | 0400 | 0427 | 0454 | 0481 | 0508 | 0536 | 27 |
| 3.69 | 1.3 0563 | 0590 | 0617 | 0644 | 0671 | 0698 | 0725 | 0752 | 0779 | 0806 | 27 |
| 3.70 | 1.3 0833 | 0860 | 0887 | 0914 | 0941 | 0968 | 0995 | 1022 | 1049 | 1076 | 27 |
| 3.71 | 1.3 1103 | 1130 | 1157 | 1184 | 1211 | 1238 | 1265 | 1292 | 1319 | 1345 | 27 |
| 3.72 | 1.3 1372 | 1399 | 1426 | 1453 | 1480 | 1507 | 1534 | 1560 | 1587 | 1614 | 27 |
| 3.73 | 1.3 1641 | 1668 | 1694 | 1721 | 1748 | 1775 | 1802 | 1828 | 1855 | 1882 | 27 |
| 3.74 | 1.3 1909 | 1935 | 1962 | 1989 | 2015 | 2042 | 2069 | 2096 | 2122 | 2149 | 27 |
| 3.75 | 1.3 2176 | 2202 | 2229 | 2256 | 2282 | 2309 | 2335 | 2362 | 2389 | 2415 | 27 |
| 3.76 | 1.3 2442 | 2468 | 2495 | 2522 | 2548 | 2575 | 2601 | 2628 | 2654 | 2681 | 27 |
| 3.77 | 1.3 2708 | 2734 | 2761 | 2787 | 2814 | 2840 | 2867 | 2893 | 2919 | 2946 | 27-26 |
| 3.78 | 1.3 2972 | 2999 | 3025 | 3052 | 3078 | 3105 | 3131 | 3157 | 3184 | 3210 | 26 |
| 3.79 | 1.3 3237 | 3263 | 3289 | 3316 | 3342 | 3368 | 3395 | 3421 | 3447 | 3474 | 26 |
| 3.80 | 1.3 3500 | 3526 | 3553 | 3579 | 3605 | 3632 | 3658 | 3684 | 3710 | 3737 | 26 |
| 3.81 | 1.3 3763 | 3789 | 3815 | 3842 | 3868 | 3894 | 3920 | 3946 | 3973 | 3999 | 26 |
| 3.82 | 1.3 4025 | 4051 | 4077 | 4104 | 4130 | 4156 | 4182 | 4208 | 4234 | 4260 | 26 |
| 3.83 | 1.3 4286 | 4313 | 4339 | 4365 | 4391 | 4417 | 4443 | 4469 | 4495 | 4521 | 26 |
| 3.84 | 1.3 4547 | 4573 | 4599 | 4625 | 4651 | 4677 | 4703 | 4729 | 4755 | 4781 | 26 |
| 3.85 | 1.3 4807 | 4833 | 4859 | 4885 | 4911 | 4937 | 4963 | 4989 | 5015 | 5041 | 26 |
| 3.86 | 1.3 5067 | 5093 | 5119 | 5144 | 5170 | 5196 | 5222 | 5248 | 5274 | 5300 | 26 |
| 3.87 | 1.3 5325 | 5351 | 5377 | 5403 | 5429 | 5455 | 5480 | 5506 | 5532 | 5558 | 26 |
| 3.88 | 1.3 5584 | 5609 | 5635 | 5661 | 5687 | 5712 | 5738 | 5764 | 5789 | 5815 | 26 |
| 3.89 | 1.3 5841 | 5867 | 5892 | 5918 | 5944 | 5969 | 5995 | 6021 | 6046 | 6072 | 26 |
| 3.90 | 1.3 6098 | 6123 | 6149 | 6175 | 6200 | 6226 | 6251 | 6277 | 6303 | 6328 | 26 |
| 3.91 | 1.3 6354 | 6379 | 6405 | 6430 | 6456 | 6481 | 6507 | 6533 | 6558 | 6584 | 26 |
| 3.92 | 1.3 6609 | 6635 | 6660 | 6686 | 6711 | 6737 | 6762 | 6788 | 6813 | 6838 | 26-25 |
| 3.93 | 1.3 6864 | 6889 | 6915 | 6940 | 6966 | 6991 | 7016 | 7042 | 7067 | 7093 | 25 |
| 3.94 | 1.3 7118 | 7143 | 7169 | 7194 | 7220 | 7245 | 7270 | 7296 | 7321 | 7346 | 25 |
| 3.95 | 1.3 7372 | 7397 | 7422 | 7447 | 7473 | 7498 | 7523 | 7549 | 7574 | 7599 | 25 |
| 3.96 | 1.3 7624 | 7650 | 7675 | 7700 | 7725 | 7751 | 7776 | 7801 | 7826 | 7851 | 25 |
| 3.97 | 1.3 7877 | 7902 | 7927 | 7952 | 7977 | 8002 | 8028 | 8053 | 8078 | 8103 | 25 |
| 3.98 | 1.3 8128 | 8143 | 8178 | 8204 | 8229 | 8254 | 8279 | 8304 | 8329 | 8354 | 25 |
| 3.99 | 1.3 8379 | 8404 | 8429 | 8454 | 8479 | 8504 | 8529 | 8554 | 8579 | 8604 | 25 |
| 4.00 | 1.3 8629 | 8654 | 8679 | 8704 | 8729 | 8754 | 8779 | 8804 | 8829 | 8854 | 25 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|------|----------|------|-------|------|------|------|-------|------|------|------|-------|
| 4.00 | 1.3 8629 | 8654 | 8679 | 8704 | 8729 | 8754 | 8779 | 8804 | 8829 | 8854 | 25 |
| 4.01 | 1.3 8879 | 8904 | 8929 | 8954 | 8979 | 9004 | 9029 | 9054 | 9078 | 9103 | 25 |
| 4.02 | 1.3 9128 | 9153 | 9178 | 9203 | 9228 | 9252 | 9277 | 9302 | 9327 | 9352 | 25 |
| 4.03 | 1.3 9377 | 9401 | 9426 | 9451 | 9476 | 9501 | 9525 | 9550 | 9575 | 9600 | 25 |
| 4.04 | 1.3 9624 | 9649 | 9674 | 9699 | 9723 | 9748 | 9773 | 9798 | 9822 | 9847 | 25 |
| 4.05 | 1.3 9872 | 9896 | 9921 | 9946 | 9970 | 9995 | *0020 | 0044 | 0069 | 0094 | 25 |
| 4.06 | 1.4 0118 | 0143 | 0168 | 0192 | 0217 | 0241 | 0266 | 0291 | 0315 | 0340 | 25 |
| 4.07 | 1.4 0364 | 0389 | 0413 | 0438 | 0463 | 0487 | 0512 | 0536 | 0561 | 0585 | 25 |
| 4.08 | 1.4 0610 | 0634 | 0659 | 0683 | 0708 | 0732 | 0757 | 0781 | 0806 | 0830 | 25-24 |
| 4.09 | 1.4 0854 | 0879 | 0903 | 0928 | 0952 | 0977 | 1001 | 1025 | 1050 | 1074 | 24 |
| 4.10 | 1.4 1099 | 1123 | 1147 | 1172 | 1196 | 1221 | 1245 | 1269 | 1294 | 1318 | 24 |
| 4.11 | 1.4 1342 | 1367 | 1391 | 1415 | 1440 | 1464 | 1488 | 1512 | 1537 | 1561 | 24 |
| 4.12 | 1.4 1585 | 1610 | 1634 | 1658 | 1682 | 1707 | 1731 | 1755 | 1779 | 1804 | 24 |
| 4.13 | 1.4 1828 | 1852 | 1876 | 1900 | 1925 | 1949 | 1973 | 1997 | 2021 | 2045 | 24 |
| 4.14 | 1.4 2070 | 2094 | 2118 | 2142 | 2166 | 2190 | 2214 | 2239 | 2263 | 2287 | 24 |
| 4.15 | 1.4 2311 | 2335 | 2359 | 2383 | 2407 | 2431 | 2455 | 2479 | 2503 | 2527 | 24 |
| 4.16 | 1.4 2552 | 2576 | 2600 | 2624 | 2648 | 2672 | 2696 | 2720 | 2744 | 2768 | 24 |
| 4.17 | 1.4 2792 | 2816 | 2840 | 2864 | 2887 | 2911 | 2935 | 2959 | 2983 | 3007 | 24 |
| 4.18 | 1.4 3031 | 3055 | 3079 | 3103 | 3127 | 3151 | 3175 | 3198 | 3222 | 3246 | 24 |
| 4.19 | 1.4 3270 | 3294 | 3318 | 3342 | 3365 | 3389 | 3413 | 3437 | 3461 | 3485 | 24 |
| 4.20 | 1.4 3508 | 3532 | 3556 | 3580 | 3604 | 3627 | 3651 | 3675 | 3699 | 3723 | 24 |
| 4.21 | 1.4 3746 | 3770 | 3794 | 3817 | 3841 | 3865 | 3889 | 3912 | 3936 | 3960 | 24 |
| 4.22 | 1.4 3984 | 4007 | 4031 | 4055 | 4078 | 4102 | 4126 | 4149 | 4173 | 4197 | 24 |
| 4.23 | 1.4 4220 | 4244 | 4267 | 4291 | 4315 | 4338 | 4362 | 4386 | 4409 | 4433 | 24 |
| 4.24 | 1.4 4456 | 4480 | 4503 | 4527 | 4551 | 4574 | 4598 | 4621 | 4645 | 4668 | 24 |
| 4.25 | 1.4 4692 | 4715 | 4739 | 4762 | 4786 | 4809 | 4833 | 4856 | 4880 | 4903 | 24-23 |
| 4.26 | 1.4 4927 | 4950 | 4974 | 4997 | 5021 | 5044 | 5068 | 5091 | 5115 | 5138 | 23 |
| 4.27 | 1.4 5161 | 5185 | 5208 | 5232 | 5255 | 5278 | 5302 | 5325 | 5349 | 5372 | 23 |
| 4.28 | 1.4 5395 | 5419 | 5442 | 5465 | 5489 | 5512 | 5535 | 5559 | 5582 | 5605 | 23 |
| 4.29 | 1.4 5629 | 5652 | 5675 | 5699 | 5722 | 5745 | 5768 | 5792 | 5815 | 5838 | 23 |
| 4.30 | 1.4 5862 | 5885 | 5908 | 5931 | 5954 | 5978 | 6001 | 6024 | 6047 | 6071 | 23 |
| 4.31 | 1.4 6094 | 6117 | 6140 | 6163 | 6187 | 6210 | 6233 | 6256 | 6279 | 6302 | 23 |
| 4.32 | 1.4 6326 | 6349 | 6372 | 6395 | 6418 | 6441 | 6464 | 6487 | 6511 | 6534 | 23 |
| 4.33 | 1.4 6557 | 6580 | 6603 | 6626 | 6649 | 6672 | 6695 | 6718 | 6741 | 6764 | 23 |
| 4.34 | 1.4 6787 | 6810 | 6834 | 6857 | 6880 | 6903 | 6926 | 6949 | 6972 | 6995 | 23 |
| 4.35 | 1.4 7018 | 7041 | 7064 | 7087 | 7109 | 7132 | 7155 | 7178 | 7201 | 7224 | 23 |
| 4.36 | 1.4 7247 | 7270 | 7293 | 7316 | 7339 | 7362 | 7385 | 7408 | 7431 | 7453 | 23 |
| 4.37 | 1.4 7476 | 7499 | 7522 | 7545 | 7568 | 7591 | 7614 | 7636 | 7659 | 7682 | 23 |
| 4.38 | 1.4 7705 | 7728 | 7751 | 7773 | 7796 | 7819 | 7842 | 7865 | 7887 | 7910 | 23 |
| 4.39 | 1.4 7933 | 7956 | 7978 | 8001 | 8024 | 8047 | 8070 | 8092 | 8115 | 8138 | 23 |
| 4.40 | 1.4 8160 | 8183 | 8206 | 8229 | 8251 | 8274 | 8297 | 8319 | 8342 | 8365 | 23 |
| 4.41 | 1.4 8387 | 8410 | 8433 | 8455 | 8478 | 8501 | 8523 | 8546 | 8569 | 8591 | 23 |
| 4.42 | 1.4 8614 | 8637 | 8659 | 8682 | 8704 | 8727 | 8750 | 8772 | 8795 | 8817 | 23 |
| 4.43 | 1.4 8840 | 8863 | 8885 | 8908 | 8930 | 8953 | 8975 | 8998 | 9020 | 9043 | 23 |
| 4.44 | 1.4 9065 | 9088 | 9110 | 9133 | 9155 | 9178 | 9200 | 9223 | 9245 | 9268 | 23 |
| 4.45 | 1.4 9290 | 9313 | 9335 | 9358 | 9380 | 9403 | 9425 | 9448 | 9470 | 9492 | 23-22 |
| 4.46 | 1.4 9515 | 9537 | 9560 | 9582 | 9605 | 9627 | 9649 | 9672 | 9694 | 9716 | 22 |
| 4.47 | 1.4 9739 | 9761 | 9784 | 9806 | 9828 | 9851 | 9873 | 9895 | 9918 | 9940 | 22 |
| 4.48 | 1.4 9962 | 9985 | *0007 | 0029 | 0052 | 0074 | 0096 | 0118 | 0141 | 0163 | 22 |
| 4.49 | 1.5 0185 | 0208 | 0230 | 0252 | 0274 | 0297 | 0319 | 0341 | 0363 | 0386 | 22 |
| 4.50 | 1.5 0408 | 0430 | 0452 | 0474 | 0497 | 0519 | 0541 | 0563 | 0585 | 0608 | 22 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|------|----------|------|------|------|-------|------|------|------|------|------|-------|
| 4.50 | 1.5 0408 | 0430 | 0452 | 0474 | 0497 | 0519 | 0541 | 0563 | 0585 | 0608 | 22 |
| 4.51 | 1.5 0630 | 0652 | 0674 | 0696 | 0718 | 0741 | 0763 | 0785 | 0807 | 0829 | 22 |
| 4.52 | 1.5 0851 | 0873 | 0895 | 0918 | 0940 | 0962 | 0984 | 1006 | 1028 | 1050 | 22 |
| 4.53 | 1.5 1072 | 1094 | 1116 | 1138 | 1160 | 1183 | 1205 | 1227 | 1249 | 1271 | 22 |
| 4.54 | 1.5 1293 | 1315 | 1337 | 1359 | 1381 | 1403 | 1425 | 1447 | 1469 | 1491 | 22 |
| 4.55 | 1.5 1513 | 1535 | 1557 | 1579 | 1601 | 1623 | 1645 | 1666 | 1688 | 1710 | 22 |
| 4.56 | 1.5 1732 | 1754 | 1776 | 1798 | 1820 | 1842 | 1864 | 1886 | 1908 | 1929 | 22 |
| 4.57 | 1.5 1951 | 1973 | 1995 | 2017 | 2039 | 2061 | 2083 | 2104 | 2126 | 2148 | 22 |
| 4.58 | 1.5 2170 | 2192 | 2214 | 2235 | 2257 | 2279 | 2301 | 2323 | 2344 | 2366 | 22 |
| 4.59 | 1.5 2388 | 2410 | 2432 | 2453 | 2475 | 2497 | 2519 | 2540 | 2562 | 2584 | 22 |
| 4.60 | 1.5 2606 | 2627 | 2649 | 2671 | 2693 | 2714 | 2736 | 2758 | 2779 | 2801 | 22 |
| 4.61 | 1.5 2823 | 2844 | 2866 | 2888 | 2910 | 2931 | 2953 | 2975 | 2996 | 3018 | 22 |
| 4.62 | 1.5 3039 | 3061 | 3083 | 3104 | 3126 | 3148 | 3169 | 3191 | 3212 | 3234 | 22 |
| 4.63 | 1.5 3256 | 3277 | 3299 | 3320 | 3342 | 3364 | 3385 | 3407 | 3428 | 3450 | 22 |
| 4.64 | 1.5 3471 | 3493 | 3515 | 3536 | 3558 | 3579 | 3601 | 3622 | 3644 | 3665 | 22 |
| 4.65 | 1.5 3687 | 3708 | 3730 | 3751 | 3773 | 3794 | 3816 | 3837 | 3859 | 3880 | 22-21 |
| 4.66 | 1.5 3902 | 3923 | 3944 | 3966 | 3987 | 4009 | 4030 | 4052 | 4073 | 4094 | 21 |
| 4.67 | 1.5 4116 | 4137 | 4159 | 4180 | 4202 | 4223 | 4244 | 4266 | 4287 | 4308 | 21 |
| 4.68 | 1.5 4330 | 4351 | 4373 | 4394 | 4415 | 4437 | 4458 | 4479 | 4501 | 4522 | 21 |
| 4.69 | 1.5 4543 | 4565 | 4586 | 4607 | 4629 | 4650 | 4671 | 4692 | 4714 | 4735 | 21 |
| 4.70 | 1.5 4756 | 4778 | 4799 | 4820 | 4841 | 4863 | 4884 | 4905 | 4926 | 4948 | 21 |
| 4.71 | 1.5 4969 | 4990 | 5011 | 5032 | 5054 | 5075 | 5096 | 5117 | 5138 | 5160 | 21 |
| 4.72 | 1.5 5181 | 5202 | 5223 | 5244 | 5266 | 5287 | 5308 | 5329 | 5350 | 5371 | 21 |
| 4.73 | 1.5 5393 | 5414 | 5435 | 5456 | 5477 | 5498 | 5519 | 5540 | 5562 | 5583 | 21 |
| 4.74 | 1.5 5604 | 5625 | 5646 | 5667 | 5688 | 5709 | 5730 | 5751 | 5772 | 5793 | 21 |
| 4.75 | 1.5 5814 | 5836 | 5857 | 5878 | 5899 | 5920 | 5941 | 5962 | 5983 | 6004 | 21 |
| 4.76 | 1.5 6025 | 6046 | 6067 | 6088 | 6109 | 6130 | 6151 | 6172 | 6193 | 6214 | 21 |
| 4.77 | 1.5 6235 | 6256 | 6277 | 6298 | 6318 | 6339 | 6360 | 6381 | 6402 | 6423 | 21 |
| 4.78 | 1.5 6444 | 6465 | 6486 | 6507 | 6528 | 6549 | 6569 | 6590 | 6611 | 6632 | 21 |
| 4.79 | 1.5 6653 | 6674 | 6695 | 6716 | 6737 | 6757 | 6778 | 6799 | 6820 | 6841 | 21 |
| 4.80 | 1.5 6862 | 6882 | 6903 | 6924 | 6945 | 6966 | 6987 | 7007 | 7028 | 7049 | 21 |
| 4.81 | 1.5 7070 | 7090 | 7111 | 7132 | 7153 | 7174 | 7194 | 7215 | 7236 | 7257 | 21 |
| 4.82 | 1.5 7277 | 7298 | 7319 | 7340 | 7360 | 7381 | 7402 | 7423 | 7443 | 7464 | 21 |
| 4.83 | 1.5 7485 | 7505 | 7526 | 7547 | 7567 | 7588 | 7609 | 7629 | 7650 | 7671 | 21 |
| 4.84 | 1.5 7691 | 7712 | 7733 | 7753 | 7774 | 7795 | 7815 | 7836 | 7857 | 7877 | 21 |
| 4.85 | 1.5 7898 | 7918 | 7939 | 7960 | 7980 | 8001 | 8022 | 8042 | 8063 | 8083 | 21 |
| 4.86 | 1.5 8104 | 8124 | 8145 | 8166 | 8186 | 8207 | 8227 | 8248 | 8268 | 8289 | 21 |
| 4.87 | 1.5 8309 | 8330 | 8350 | 8371 | 8391 | 8412 | 8433 | 8453 | 8474 | 8494 | 21-20 |
| 4.88 | 1.5 8515 | 8535 | 8555 | 8576 | 8596 | 8617 | 8637 | 8658 | 8678 | 8699 | 20 |
| 4.89 | 1.5 8719 | 8740 | 8760 | 8781 | 8801 | 8821 | 8842 | 8862 | 8883 | 8903 | 20 |
| 4.90 | 1.5 8924 | 8944 | 8964 | 8985 | 9005 | 9026 | 9046 | 9066 | 9087 | 9107 | 20 |
| 4.91 | 1.5 9127 | 9148 | 9168 | 9188 | 9209 | 9229 | 9250 | 9270 | 9290 | 9311 | 20 |
| 4.92 | 1.5 9331 | 9351 | 9371 | 9392 | 9412 | 9432 | 9453 | 9473 | 9493 | 9514 | 20 |
| 4.93 | 1.5 9534 | 9554 | 9574 | 9595 | 9615 | 9635 | 9656 | 9676 | 9696 | 9716 | 20 |
| 4.94 | 1.5 9737 | 9757 | 9777 | 9797 | 9817 | 9838 | 9858 | 9878 | 9898 | 9919 | 20 |
| 4.95 | 1.5 9939 | 9959 | 9979 | 9999 | *0020 | 0040 | 0060 | 0080 | 0100 | 0120 | 20 |
| 4.96 | 1.6 0141 | 0161 | 0181 | 0201 | 0221 | 0241 | 0261 | 0282 | 0302 | 0322 | 20 |
| 4.97 | 1.6 0342 | 0362 | 0382 | 0402 | 0422 | 0443 | 0463 | 0483 | 0503 | 0523 | 20 |
| 4.98 | 1.6 0543 | 0563 | 0583 | 0603 | 0623 | 0643 | 0663 | 0683 | 0704 | 0724 | 20 |
| 4.99 | 1.6 0744 | 0764 | 0784 | 0804 | 0824 | 0844 | 0864 | 0884 | 0904 | 0924 | 20 |
| 5.00 | 1.6 0944 | 0964 | 0984 | 1004 | 1024 | 1044 | 1064 | 1084 | 1104 | 1124 | 20 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

Five-Place Natural Logarithms.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|------|---------|------|------|-------|------|-------|------|-------|-------|-------|---------|
| 5.0 | 1.60944 | 1144 | 1343 | 1542 | 1741 | 1939 | 2137 | 2334 | 2531 | 2728 | 200-196 |
| 5.1 | 1.62924 | 3120 | 3315 | 3511 | 3705 | 3900 | 4094 | 4287 | 4481 | 4673 | 196-192 |
| 5.2 | 1.64866 | 5058 | 5250 | 5441 | 5632 | 5823 | 6013 | 6203 | 6393 | 6582 | 192-189 |
| 5.3 | 1.66771 | 6959 | 7147 | 7335 | 7523 | 7710 | 7896 | 8083 | 8269 | 8455 | 189-185 |
| 5.4 | 1.68640 | 8825 | 9010 | 9194 | 9378 | 9562 | 9745 | 9928 | *0111 | 0293 | 185-182 |
| 5.5 | 1.70475 | 0656 | 0838 | 1019 | 1199 | 1380 | 1560 | 1740 | 1919 | 2098 | 182-179 |
| 5.6 | 1.72277 | 2455 | 2633 | 2811 | 2988 | 3166 | 3342 | 3519 | 3695 | 3871 | 178-176 |
| 5.7 | 1.74047 | 4222 | 4397 | 4572 | 4746 | 4920 | 5094 | 5267 | 5440 | 5613 | 175-173 |
| 5.8 | 1.75786 | 5958 | 6130 | 6302 | 6473 | 6644 | 6815 | 6985 | 7156 | 7326 | 172-170 |
| 5.9 | 1.77495 | 7665 | 7834 | 8002 | 8171 | 8339 | 8507 | 8675 | 8842 | 9009 | 169-167 |
| 6.0 | 1.79176 | 9342 | 9509 | 9675 | 9840 | *0006 | 0171 | 0336 | 0500 | 0665 | 167-164 |
| 6.1 | 1.80829 | 0993 | 1156 | 1319 | 1482 | 1645 | 1808 | 1970 | 2132 | 2294 | 164-161 |
| 6.2 | 1.82455 | 2616 | 2777 | 2938 | 3098 | 3258 | 3418 | 3578 | 3737 | 3896 | 161-159 |
| 6.3 | 1.84055 | 4214 | 4372 | 4530 | 4688 | 4845 | 5003 | 5160 | 5317 | 5473 | 159-156 |
| 6.4 | 1.85630 | 5786 | 5942 | 6097 | 6253 | 6408 | 6563 | 6718 | 6872 | 7026 | 156-154 |
| 6.5 | 1.87180 | 7334 | 7487 | 7641 | 7794 | 7947 | 8099 | 8251 | 8403 | 8555 | 154-152 |
| 6.6 | 1.88707 | 8858 | 9010 | 9160 | 9311 | 9462 | 9612 | 9762 | 9912 | *0061 | 151-149 |
| 6.7 | 1.90211 | 0360 | 0509 | 0658 | 0806 | 0954 | 1102 | 1250 | 1398 | 1545 | 149-147 |
| 6.8 | 1.91692 | 1839 | 1986 | 2132 | 2279 | 2425 | 2571 | 2716 | 2862 | 3007 | 147-145 |
| 6.9 | 1.93152 | 3297 | 3442 | 3586 | 3730 | 3874 | 4018 | 4162 | 4305 | 4448 | 145-143 |
| 7.0 | 1.94591 | 4734 | 4876 | 5019 | 5161 | 5303 | 5445 | 5586 | 5727 | 5869 | 143-141 |
| 7.1 | 1.96009 | 6150 | 6291 | 6431 | 6571 | 6711 | 6851 | 6991 | 7130 | 7269 | 141-139 |
| 7.2 | 1.97408 | 7547 | 7685 | 7824 | 7962 | 8100 | 8238 | 8376 | 8513 | 8650 | 139-137 |
| 7.3 | 1.98787 | 8924 | 9061 | 9198 | 9334 | 9470 | 9606 | 9742 | 9877 | *0013 | 137-135 |
| 7.4 | 2.00148 | 0283 | 0418 | 0553 | 0687 | 0821 | 0956 | 1089 | 1223 | 1357 | 135-133 |
| 7.5 | 2.01490 | 1624 | 1757 | 1890 | 2022 | 2155 | 2287 | 2419 | 2551 | 2683 | 133-132 |
| 7.6 | 2.02815 | 2946 | 3078 | 3209 | 3340 | 3471 | 3601 | 3732 | 3862 | 3992 | 131-130 |
| 7.7 | 2.04122 | 4252 | 4381 | 4511 | 4640 | 4769 | 4898 | 5027 | 5156 | 5284 | 130-128 |
| 7.8 | 2.05412 | 5540 | 5668 | 5796 | 5924 | 6051 | 6179 | 6306 | 6433 | 6560 | 128-127 |
| 7.9 | 2.06686 | 6813 | 6939 | 7065 | 7191 | 7317 | 7443 | 7568 | 7694 | 7819 | 127-125 |
| 8.0 | 2.07944 | 8069 | 8194 | 8318 | 8443 | 8567 | 8691 | 8815 | 8939 | 9063 | 125-124 |
| 8.1 | 2.09186 | 9310 | 9433 | 9556 | 9679 | 9802 | 9924 | *0047 | 0169 | 0291 | 123-122 |
| 8.2 | 2.10413 | 0535 | 0657 | 0779 | 0900 | 1021 | 1142 | 1263 | 1384 | 1505 | 122-121 |
| 8.3 | 2.11626 | 1746 | 1866 | 1986 | 2106 | 2226 | 2346 | 2465 | 2585 | 2704 | 120-119 |
| 8.4 | 2.12823 | 2942 | 3061 | 3180 | 3298 | 3417 | 3535 | 3653 | 3771 | 3889 | 119-118 |
| 8.5 | 2.14007 | 4124 | 4242 | 4359 | 4476 | 4593 | 4710 | 4827 | 4943 | 5060 | 118-116 |
| 8.6 | 2.15176 | 5292 | 5409 | 5524 | 5640 | 5756 | 5871 | 5987 | 6102 | 6217 | 116-115 |
| 8.7 | 2.16332 | 6447 | 6562 | 6677 | 6791 | 6905 | 7020 | 7134 | 7248 | 7361 | 115-114 |
| 8.8 | 2.17475 | 7589 | 7702 | 7816 | 7929 | 8042 | 8155 | 8267 | 8380 | 8493 | 114-112 |
| 8.9 | 2.18605 | 8717 | 8830 | 8942 | 9054 | 9165 | 9277 | 9389 | 9500 | 9611 | 112-111 |
| 9.0 | 2.19722 | 9834 | 9944 | *0055 | 0166 | 0276 | 0387 | 0497 | 0607 | 0717 | 111-110 |
| 9.1 | 2.20827 | 0937 | 1047 | 1157 | 1266 | 1375 | 1485 | 1594 | 1703 | 1812 | 110-109 |
| 9.2 | 2.21920 | 2029 | 2138 | 2246 | 2354 | 2462 | 2570 | 2678 | 2786 | 2894 | 109-108 |
| 9.3 | 2.23001 | 3109 | 3216 | 3324 | 3431 | 3538 | 3645 | 3751 | 3858 | 3965 | 107-106 |
| 9.4 | 2.24071 | 4177 | 4284 | 4390 | 4496 | 4601 | 4707 | 4813 | 4918 | 5024 | 106-105 |
| 9.5 | 2.25129 | 5234 | 5339 | 5444 | 5549 | 5654 | 5759 | 5863 | 5968 | 6072 | 105-104 |
| 9.6 | 2.26176 | 6280 | 6384 | 6488 | 6592 | 6696 | 6799 | 6903 | 7006 | 7109 | 104-103 |
| 9.7 | 2.27213 | 7316 | 7419 | 7521 | 7624 | 7727 | 7829 | 7932 | 8034 | 8136 | 103-102 |
| 9.8 | 2.28238 | 8340 | 8442 | 8544 | 8646 | 8747 | 8849 | 8950 | 9051 | 9152 | 102-101 |
| 9.9 | 2.29253 | 9354 | 9455 | 9556 | 9657 | 9757 | 9858 | 9958 | *0058 | 0158 | 101-100 |
| 10.0 | 2.30259 | 0358 | 0458 | 0558 | 0658 | 0757 | 0857 | 0956 | 1055 | 1154 | 100-99 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

The Natural Logarithms (each increased by 10.) of Numbers between 0.00 and 0.99.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.0 | | 5.395 | 6.088 | 6.493 | 6.781 | 7.004 | 7.187 | 7.341 | 7.474 | 7.592 |
| 0.1 | 7.697 | 7.793 | 7.880 | 7.960 | 8.034 | 8.103 | 8.167 | 8.228 | 8.285 | 8.339 |
| 0.2 | 8.391 | 8.439 | 8.486 | 8.530 | 8.573 | 8.614 | 8.653 | 8.691 | 8.727 | 8.762 |
| 0.3 | 8.796 | 8.829 | 8.861 | 8.891 | 8.921 | 8.950 | 8.978 | 9.006 | 9.032 | 9.058 |
| 0.4 | 9.084 | 9.108 | 9.132 | 9.156 | 9.179 | 9.201 | 9.223 | 9.245 | 9.266 | 9.287 |
| 0.5 | 9.307 | 9.327 | 9.346 | 9.365 | 9.384 | 9.402 | 9.420 | 9.438 | 9.455 | 9.472 |
| 0.6 | 9.489 | 9.506 | 9.522 | 9.538 | 9.554 | 9.569 | 9.584 | 9.600 | 9.614 | 9.629 |
| 0.7 | 9.643 | 9.658 | 9.671 | 9.685 | 9.699 | 9.712 | 9.726 | 9.739 | 9.752 | 9.764 |
| 0.8 | 9.777 | 9.789 | 9.802 | 9.814 | 9.826 | 9.837 | 9.849 | 9.861 | 9.872 | 9.883 |
| 0.9 | 9.895 | 9.906 | 9.917 | 9.927 | 9.938 | 9.949 | 9.959 | 9.970 | 9.980 | 9.990 |

Note : $\log_e x = \log_{10} x \cdot \log_e 10 = (2.30259) \log_{10} x$.

The Natural Logarithms of Whole Numbers from 10 to 209.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|--------|-------|------|------|------|-------|------|------|------|-------|
| 1 | 2.3026 | 3979 | 4849 | 5649 | 6391 | 7080 | 7726 | 8332 | 8904 | 9444 |
| 2 | 2.9957 | *0445 | 0910 | 1355 | 1781 | 2189 | 2581 | 2958 | 3322 | 3673 |
| 3 | 3.4012 | 4340 | 4657 | 4965 | 5264 | 5553 | 5835 | 6109 | 6376 | 6636 |
| 4 | 3.6889 | 7136 | 7377 | 7612 | 7842 | 8067 | 8286 | 8501 | 8712 | 8918 |
| 5 | 3.9120 | 9318 | 9512 | 9703 | 9890 | *0073 | 0254 | 0431 | 0604 | 0775 |
| 6 | 4.0943 | 1109 | 1271 | 1431 | 1589 | 1744 | 1897 | 2047 | 2195 | 2341 |
| 7 | 4.2485 | 2627 | 2767 | 2905 | 3041 | 3175 | 3307 | 3438 | 3567 | 3694 |
| 8 | 4.3820 | 3944 | 4067 | 4188 | 4308 | 4427 | 4543 | 4659 | 4773 | 4886 |
| 9 | 4.4998 | 5109 | 5218 | 5326 | 5433 | 5539 | 5643 | 5747 | 5850 | 5951 |
| 10 | 4.6052 | 6151 | 6250 | 6347 | 6444 | 6540 | 6634 | 6728 | 6821 | 6913 |
| 11 | 4.7005 | 7095 | 7185 | 7274 | 7362 | 7449 | 7536 | 7622 | 7707 | 7791 |
| 12 | 4.7875 | 7958 | 8040 | 8122 | 8203 | 8283 | 8363 | 8442 | 8520 | 8598 |
| 13 | 4.8675 | 8752 | 8828 | 8903 | 8978 | 9053 | 9127 | 9200 | 9273 | 9345 |
| 14 | 4.9416 | 9488 | 9558 | 9628 | 9698 | 9767 | 9836 | 9904 | 9972 | *0039 |
| 15 | 5.0106 | 0173 | 0239 | 0304 | 0370 | 0434 | 0499 | 0562 | 0626 | 0689 |
| 16 | 5.0752 | 0814 | 0876 | 0938 | 0999 | 1059 | 1120 | 1180 | 1240 | 1299 |
| 17 | 5.1358 | 1417 | 1475 | 1533 | 1591 | 1648 | 1705 | 1762 | 1818 | 1874 |
| 18 | 5.1930 | 1985 | 2040 | 2095 | 2149 | 2204 | 2257 | 2311 | 2364 | 2417 |
| 19 | 5.2470 | 2523 | 2575 | 2627 | 2679 | 2730 | 2781 | 2832 | 2883 | 2933 |
| 20 | 5.2983 | 3033 | 3083 | 3132 | 3181 | 3230 | 3279 | 3327 | 3375 | 3423 |

Note : $\log_e 10 = 2.30258509$.

$\log_e 100 = 4.60517019$.

The Common Logarithms of $\Gamma(n)$ for Values of n between 1 and 2.

$$\Gamma(n) = \int_0^{\infty} x^{n-1} \cdot e^{-x} dx = \int_0^1 \left[\log \frac{1}{x} \right]^{n-1} dx.$$

| n | $\log_{10} \Gamma(n)$ | n | $\log_{10} \Gamma(n)$ | n | $\log_{10} \Gamma(n)$ | n | $\log_{10} \Gamma(n)$ | n | $\log_{10} \Gamma(n)$ |
|------|-----------------------|------|-----------------------|------|-----------------------|------|-----------------------|------|-----------------------|
| 1.01 | 1.9975 | 1.21 | 1.9617 | 1.41 | 1.9478 | 1.61 | 1.9517 | 1.81 | 1.9704 |
| 1.02 | 1.9951 | 1.22 | 1.9605 | 1.42 | 1.9476 | 1.62 | 1.9523 | 1.82 | 1.9717 |
| 1.03 | 1.9928 | 1.23 | 1.9594 | 1.43 | 1.9475 | 1.63 | 1.9529 | 1.83 | 1.9730 |
| 1.04 | 1.9905 | 1.24 | 1.9583 | 1.44 | 1.9473 | 1.64 | 1.9536 | 1.84 | 1.9743 |
| 1.05 | 1.9883 | 1.25 | 1.9573 | 1.45 | 1.9473 | 1.65 | 1.9543 | 1.85 | 1.9757 |
| 1.06 | 1.9862 | 1.26 | 1.9564 | 1.46 | 1.9472 | 1.66 | 1.9550 | 1.86 | 1.9771 |
| 1.07 | 1.9841 | 1.27 | 1.9554 | 1.47 | 1.9473 | 1.67 | 1.9558 | 1.87 | 1.9786 |
| 1.08 | 1.9821 | 1.28 | 1.9546 | 1.48 | 1.9473 | 1.68 | 1.9566 | 1.88 | 1.9800 |
| 1.09 | 1.9802 | 1.29 | 1.9538 | 1.49 | 1.9474 | 1.69 | 1.9575 | 1.89 | 1.9815 |
| 1.10 | 1.9783 | 1.30 | 1.9530 | 1.50 | 1.9475 | 1.70 | 1.9584 | 1.90 | 1.9831 |
| 1.11 | 1.9765 | 1.31 | 1.9523 | 1.51 | 1.9477 | 1.71 | 1.9593 | 1.91 | 1.9846 |
| 1.12 | 1.9748 | 1.32 | 1.9516 | 1.52 | 1.9479 | 1.72 | 1.9603 | 1.92 | 1.9862 |
| 1.13 | 1.9731 | 1.33 | 1.9510 | 1.53 | 1.9482 | 1.73 | 1.9613 | 1.93 | 1.9878 |
| 1.14 | 1.9715 | 1.34 | 1.9505 | 1.54 | 1.9485 | 1.74 | 1.9623 | 1.94 | 1.9895 |
| 1.15 | 1.9699 | 1.35 | 1.9500 | 1.55 | 1.9488 | 1.75 | 1.9633 | 1.95 | 1.9912 |
| 1.16 | 1.9684 | 1.36 | 1.9495 | 1.56 | 1.9492 | 1.76 | 1.9644 | 1.96 | 1.9929 |
| 1.17 | 1.9669 | 1.37 | 1.9491 | 1.57 | 1.9496 | 1.77 | 1.9656 | 1.97 | 1.9946 |
| 1.18 | 1.9655 | 1.38 | 1.9487 | 1.58 | 1.9501 | 1.78 | 1.9667 | 1.98 | 1.9964 |
| 1.19 | 1.9642 | 1.39 | 1.9483 | 1.59 | 1.9506 | 1.79 | 1.9679 | 1.99 | 1.9982 |
| 1.20 | 1.9629 | 1.40 | 1.9481 | 1.60 | 1.9511 | 1.80 | 1.9691 | 2.00 | 0.0000 |

$$\Gamma(z+1) = z \cdot \Gamma(z), \quad z > 1.$$

NATURAL TRIGONOMETRIC FUNCTIONS.

| Angle. | Sin. | Csc. | Tan. | Ctn. | Sec. | Cos. | |
|--------|-------|-------|-------|-------|-------|-------|--------|
| 0° | 0.000 | ∞ | 0.000 | ∞ | 1.000 | 1.000 | 90° |
| 1 | 0.017 | 57.30 | 0.017 | 57.29 | 1.000 | 1.000 | 89 |
| 2 | 0.035 | 28.65 | 0.035 | 28.64 | 1.001 | 0.999 | 88 |
| 3 | 0.052 | 19.11 | 0.052 | 19.08 | 1.001 | 0.999 | 87 |
| 4 | 0.070 | 14.34 | 0.070 | 14.30 | 1.002 | 0.998 | 86 |
| 5° | 0.087 | 11.47 | 0.087 | 11.43 | 1.004 | 0.996 | 85° |
| 6 | 0.105 | 9.567 | 0.105 | 9.514 | 1.006 | 0.995 | 84 |
| 7 | 0.122 | 8.206 | 0.123 | 8.144 | 1.008 | 0.993 | 83 |
| 8 | 0.139 | 7.185 | 0.141 | 7.115 | 1.010 | 0.990 | 82 |
| 9 | 0.156 | 6.392 | 0.158 | 6.314 | 1.012 | 0.988 | 81 |
| 10° | 0.174 | 5.759 | 0.176 | 5.671 | 1.015 | 0.985 | 80° |
| 11 | 0.191 | 5.241 | 0.194 | 5.145 | 1.019 | 0.982 | 79 |
| 12 | 0.208 | 4.810 | 0.213 | 4.705 | 1.022 | 0.978 | 78 |
| 13 | 0.225 | 4.445 | 0.231 | 4.331 | 1.026 | 0.974 | 77 |
| 14 | 0.242 | 4.134 | 0.249 | 4.011 | 1.031 | 0.970 | 76 |
| 15° | 0.259 | 3.864 | 0.268 | 3.732 | 1.035 | 0.966 | 75° |
| 16 | 0.276 | 3.628 | 0.287 | 3.487 | 1.040 | 0.961 | 74 |
| 17 | 0.292 | 3.420 | 0.306 | 3.271 | 1.046 | 0.956 | 73 |
| 18 | 0.309 | 3.236 | 0.325 | 3.078 | 1.051 | 0.951 | 72 |
| 19 | 0.326 | 3.072 | 0.344 | 2.904 | 1.058 | 0.946 | 71 |
| 20° | 0.342 | 2.924 | 0.364 | 2.747 | 1.064 | 0.940 | 70° |
| 21 | 0.358 | 2.790 | 0.384 | 2.605 | 1.071 | 0.934 | 69 |
| 22 | 0.375 | 2.669 | 0.404 | 2.475 | 1.079 | 0.927 | 68 |
| 23 | 0.391 | 2.559 | 0.424 | 2.356 | 1.086 | 0.921 | 67 |
| 24 | 0.407 | 2.459 | 0.445 | 2.246 | 1.095 | 0.914 | 66 |
| 25° | 0.423 | 2.366 | 0.466 | 2.145 | 1.103 | 0.906 | 65° |
| 26 | 0.438 | 2.281 | 0.488 | 2.050 | 1.113 | 0.899 | 64 |
| 27 | 0.454 | 2.203 | 0.510 | 1.963 | 1.122 | 0.891 | 63 |
| 28 | 0.469 | 2.130 | 0.532 | 1.881 | 1.133 | 0.883 | 62 |
| 29 | 0.485 | 2.063 | 0.554 | 1.804 | 1.143 | 0.875 | 61 |
| 30° | 0.500 | 2.000 | 0.577 | 1.732 | 1.155 | 0.866 | 60° |
| 31 | 0.515 | 1.942 | 0.601 | 1.664 | 1.167 | 0.857 | 59 |
| 32 | 0.530 | 1.887 | 0.625 | 1.600 | 1.179 | 0.848 | 58 |
| 33 | 0.545 | 1.836 | 0.649 | 1.540 | 1.192 | 0.839 | 57 |
| 34 | 0.559 | 1.788 | 0.675 | 1.483 | 1.206 | 0.829 | 56 |
| 35° | 0.574 | 1.743 | 0.700 | 1.428 | 1.221 | 0.819 | 55° |
| 36 | 0.588 | 1.701 | 0.727 | 1.376 | 1.236 | 0.809 | 54 |
| 37 | 0.602 | 1.662 | 0.754 | 1.327 | 1.252 | 0.799 | 53 |
| 38 | 0.616 | 1.624 | 0.781 | 1.280 | 1.269 | 0.788 | 52 |
| 39 | 0.629 | 1.589 | 0.810 | 1.235 | 1.287 | 0.777 | 51 |
| 40° | 0.643 | 1.556 | 0.839 | 1.192 | 1.305 | 0.766 | 50° |
| 41 | 0.656 | 1.524 | 0.869 | 1.150 | 1.325 | 0.755 | 49 |
| 42 | 0.669 | 1.494 | 0.900 | 1.111 | 1.346 | 0.743 | 48 |
| 43 | 0.682 | 1.466 | 0.933 | 1.072 | 1.367 | 0.731 | 47 |
| 44 | 0.695 | 1.440 | 0.966 | 1.036 | 1.390 | 0.719 | 46 |
| 45° | 0.707 | 1.414 | 1.000 | 1.000 | 1.414 | 0.707 | 45° |
| | Cos. | Sec. | Ctn. | Tan. | Csc. | Sin. | Angle. |

Logarithms.

| N | | | | | | | | | | | P. P. |
|----|------|------|------|------|------|------|------|------|------|------|---------------|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1. 2. 3. 4. 5 |
| 10 | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4. 8.12-17.21 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4. 8.11-15.19 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1038 | 1072 | 1106 | 3. 7.10.14-17 |
| 13 | 1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3. 6.10-13.16 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3. 6. 9.12.15 |
| 15 | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3. 6. 8.11.14 |
| 16 | 2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3. 5. 8.11.13 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 2. 5. 7.10.12 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2. 5. 7. 9.12 |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2. 4. 7. 9.11 |
| 20 | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2. 4. 6. 8.11 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2. 4. 6. 8.10 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2. 4. 6. 8.10 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2. 4. 5. 7. 9 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2. 4. 5. 7. 9 |
| 25 | 3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4110 | 4133 | 2. 3. 5. 7. 9 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2. 3. 5. 7. 8 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2. 3. 5. 6. 8 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2. 3. 5. 6. 8 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1. 3. 4. 6. 7 |
| 30 | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1. 3. 4. 6. 7 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1. 3. 4. 6. 7 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1. 3. 4. 5. 7 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1. 3. 4. 5. 6 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1. 3. 4. 5. 6 |
| 35 | 5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1. 2. 4. 5. 6 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1. 2. 4. 5. 6 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1. 2. 3. 5. 6 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1. 2. 3. 5. 6 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1. 2. 3. 4. 6 |
| 40 | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1. 2. 3. 4. 5 |
| 41 | 6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1. 2. 3. 4. 5 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1. 2. 3. 4. 5 |
| 43 | 6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1. 2. 3. 4. 5 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1. 2. 3. 4. 5 |
| 45 | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1. 2. 3. 4. 5 |
| 46 | 6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1. 2. 3. 4. 5 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1. 2. 3. 4. 5 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1. 2. 3. 4. 4 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1. 2. 3. 4. 4 |
| 50 | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1. 2. 3. 3. 4 |
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1. 2. 3. 3. 4 |
| 52 | 7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1. 2. 2. 3. 4 |
| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1. 2. 2. 3. 4 |
| 54 | 7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1. 2. 2. 3. 4 |

NOTE.—This page and the three that follow it are taken from the Mathematical Tables of Prof. J. M. Peirce, published by Messrs. Ginn & Co.

Logarithms.

| N | | | | | | | | | | | P. P. |
|-----------|------|------|------|------|------|------|------|------|------|------|---------------|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1. 2. 3. 4. 5 |
| 55 | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1. 2. 2. 3. 4 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1. 2. 2. 3. 4 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1. 2. 2. 3. 4 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1. 1. 2. 3. 4 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1. 1. 2. 3. 4 |
| 60 | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1. 1. 2. 3. 4 |
| 61 | 7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1. 1. 2. 3. 4 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1. 1. 2. 3. 3 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1. 1. 2. 3. 3 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1. 1. 2. 3. 3 |
| 65 | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1. 1. 2. 3. 3 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1. 1. 2. 3. 3 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1. 1. 2. 3. 3 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1. 1. 2. 3. 3 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1. 1. 2. 3. 3 |
| 70 | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1. 1. 2. 2. 3 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1. 1. 2. 2. 3 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1. 1. 2. 2. 3 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1. 1. 2. 2. 3 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1. 1. 2. 2. 3 |
| 75 | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1. 1. 2. 2. 3 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1. 1. 2. 2. 3 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1. 1. 2. 2. 3 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1. 1. 2. 2. 3 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1. 1. 2. 2. 3 |
| 80 | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1. 1. 2. 2. 3 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1. 1. 2. 2. 3 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1. 1. 2. 2. 3 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1. 1. 2. 2. 3 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1. 1. 2. 2. 3 |
| 85 | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1. 1. 2. 2. 3 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1. 1. 2. 2. 3 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0. 1. 1. 2. 2 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0. 1. 1. 2. 2 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0. 1. 1. 2. 2 |
| 90 | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0. 1. 1. 2. 2 |
| 91 | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0. 1. 1. 2. 2 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0. 1. 1. 2. 2 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0. 1. 1. 2. 2 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0. 1. 1. 2. 2 |
| 95 | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0. 1. 1. 2. 2 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0. 1. 1. 2. 2 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0. 1. 1. 2. 2 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0. 1. 1. 2. 2 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0. 1. 1. 2. 2 |

Logarithms.

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|------------|------|------|------|------|------|------|------|------|------|------|------|
| 100 | 0000 | 0004 | 0009 | 0013 | 0017 | 0022 | 0026 | 0030 | 0035 | 0039 | 0043 |
| 101 | 0043 | 0043 | 0052 | 0056 | 0060 | 0065 | 0069 | 0073 | 0077 | 0082 | 0086 |
| 102 | 0086 | 0090 | 0095 | 0099 | 0103 | 0107 | 0111 | 0116 | 0120 | 0124 | 0128 |
| 103 | 0128 | 0133 | 0137 | 0141 | 0145 | 0149 | 0154 | 0158 | 0162 | 0166 | 0170 |
| 104 | 0170 | 0175 | 0179 | 0183 | 0187 | 0191 | 0195 | 0199 | 0204 | 0208 | 0212 |
| 105 | 0212 | 0216 | 0220 | 0224 | 0228 | 0233 | 0237 | 0241 | 0245 | 0249 | 0253 |
| 106 | 0253 | 0257 | 0261 | 0265 | 0269 | 0273 | 0278 | 0282 | 0286 | 0290 | 0294 |
| 107 | 0294 | 0298 | 0302 | 0306 | 0310 | 0314 | 0318 | 0322 | 0326 | 0330 | 0334 |
| 108 | 0334 | 0338 | 0342 | 0346 | 0350 | 0354 | 0358 | 0362 | 0366 | 0370 | 0374 |
| 109 | 0374 | 0378 | 0382 | 0386 | 0390 | 0394 | 0398 | 0402 | 0406 | 0410 | 0414 |
| 110 | 0414 | 0418 | 0422 | 0426 | 0430 | 0434 | 0438 | 0441 | 0445 | 0449 | 0453 |
| 111 | 0453 | 0457 | 0461 | 0465 | 0469 | 0473 | 0477 | 0481 | 0484 | 0488 | 0492 |
| 112 | 0492 | 0496 | 0500 | 0504 | 0508 | 0512 | 0515 | 0519 | 0523 | 0527 | 0531 |
| 113 | 0531 | 0535 | 0538 | 0542 | 0546 | 0550 | 0554 | 0558 | 0561 | 0565 | 0569 |
| 114 | 0569 | 0573 | 0577 | 0580 | 0584 | 0588 | 0592 | 0596 | 0599 | 0603 | 0607 |
| 115 | 0607 | 0611 | 0615 | 0618 | 0622 | 0626 | 0630 | 0633 | 0637 | 0641 | 0645 |
| 116 | 0645 | 0648 | 0652 | 0656 | 0660 | 0663 | 0667 | 0671 | 0674 | 0678 | 0682 |
| 117 | 0682 | 0686 | 0689 | 0693 | 0697 | 0700 | 0704 | 0708 | 0711 | 0715 | 0719 |
| 118 | 0719 | 0722 | 0726 | 0730 | 0734 | 0737 | 0741 | 0745 | 0748 | 0752 | 0756 |
| 119 | 0755 | 0759 | 0763 | 0766 | 0770 | 0774 | 0777 | 0781 | 0785 | 0788 | 0792 |
| 120 | 0792 | 0795 | 0799 | 0803 | 0806 | 0810 | 0813 | 0817 | 0821 | 0824 | 0828 |
| 121 | 0828 | 0831 | 0835 | 0839 | 0842 | 0846 | 0849 | 0853 | 0856 | 0860 | 0864 |
| 122 | 0864 | 0867 | 0871 | 0874 | 0878 | 0881 | 0885 | 0888 | 0892 | 0896 | 0899 |
| 123 | 0899 | 0903 | 0906 | 0910 | 0913 | 0917 | 0920 | 0924 | 0927 | 0931 | 0934 |
| 124 | 0934 | 0938 | 0941 | 0945 | 0948 | 0952 | 0955 | 0959 | 0962 | 0966 | 0969 |
| 125 | 0969 | 0973 | 0976 | 0980 | 0983 | 0986 | 0990 | 0993 | 0997 | 1000 | 1004 |
| 126 | 1004 | 1007 | 1011 | 1014 | 1017 | 1021 | 1024 | 1028 | 1031 | 1035 | 1038 |
| 127 | 1038 | 1041 | 1045 | 1048 | 1052 | 1055 | 1059 | 1062 | 1065 | 1069 | 1072 |
| 128 | 1072 | 1075 | 1079 | 1082 | 1086 | 1089 | 1092 | 1096 | 1099 | 1103 | 1106 |
| 129 | 1106 | 1109 | 1113 | 1116 | 1119 | 1123 | 1126 | 1129 | 1133 | 1136 | 1139 |
| 130 | 1139 | 1143 | 1146 | 1149 | 1153 | 1156 | 1159 | 1163 | 1166 | 1169 | 1173 |
| 131 | 1173 | 1176 | 1179 | 1183 | 1186 | 1189 | 1193 | 1196 | 1199 | 1202 | 1206 |
| 132 | 1206 | 1209 | 1212 | 1216 | 1219 | 1222 | 1225 | 1229 | 1232 | 1235 | 1239 |
| 133 | 1239 | 1242 | 1245 | 1248 | 1252 | 1255 | 1258 | 1261 | 1265 | 1268 | 1271 |
| 134 | 1271 | 1274 | 1278 | 1281 | 1284 | 1287 | 1290 | 1294 | 1297 | 1300 | 1303 |
| 135 | 1303 | 1307 | 1310 | 1313 | 1316 | 1319 | 1323 | 1326 | 1329 | 1332 | 1335 |
| 136 | 1335 | 1339 | 1342 | 1345 | 1348 | 1351 | 1355 | 1358 | 1361 | 1364 | 1367 |
| 137 | 1367 | 1370 | 1374 | 1377 | 1380 | 1383 | 1386 | 1389 | 1392 | 1396 | 1399 |
| 138 | 1399 | 1402 | 1405 | 1408 | 1411 | 1414 | 1418 | 1421 | 1424 | 1427 | 1430 |
| 139 | 1430 | 1433 | 1436 | 1440 | 1443 | 1446 | 1449 | 1452 | 1455 | 1458 | 1461 |
| 140 | 1461 | 1464 | 1467 | 1471 | 1474 | 1477 | 1480 | 1483 | 1486 | 1489 | 1492 |
| 141 | 1492 | 1495 | 1498 | 1501 | 1504 | 1508 | 1511 | 1514 | 1517 | 1520 | 1523 |
| 142 | 1523 | 1526 | 1529 | 1532 | 1535 | 1538 | 1541 | 1544 | 1547 | 1550 | 1553 |
| 143 | 1553 | 1556 | 1559 | 1562 | 1565 | 1569 | 1572 | 1575 | 1578 | 1581 | 1584 |
| 144 | 1584 | 1587 | 1590 | 1593 | 1596 | 1599 | 1602 | 1605 | 1608 | 1611 | 1614 |
| 145 | 1614 | 1617 | 1620 | 1623 | 1626 | 1629 | 1632 | 1635 | 1638 | 1641 | 1644 |
| 146 | 1644 | 1647 | 1649 | 1652 | 1655 | 1658 | 1661 | 1664 | 1667 | 1670 | 1673 |
| 147 | 1673 | 1676 | 1679 | 1682 | 1685 | 1688 | 1691 | 1694 | 1697 | 1700 | 1703 |
| 148 | 1703 | 1706 | 1708 | 1711 | 1714 | 1717 | 1720 | 1723 | 1726 | 1729 | 1732 |
| 149 | 1732 | 1735 | 1738 | 1741 | 1744 | 1746 | 1749 | 1752 | 1755 | 1758 | 1761 |

Logarithms.

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|------------|------|------|------|------|------|------|------|------|------|------|------|
| 150 | 1761 | 1764 | 1767 | 1770 | 1772 | 1775 | 1778 | 1781 | 1784 | 1787 | 1790 |
| 151 | 1790 | 1793 | 1796 | 1798 | 1801 | 1804 | 1807 | 1810 | 1813 | 1816 | 1818 |
| 152 | 1818 | 1821 | 1824 | 1827 | 1830 | 1833 | 1836 | 1838 | 1841 | 1844 | 1847 |
| 153 | 1847 | 1850 | 1853 | 1855 | 1858 | 1861 | 1864 | 1867 | 1870 | 1872 | 1875 |
| 154 | 1875 | 1878 | 1881 | 1884 | 1886 | 1889 | 1892 | 1895 | 1898 | 1901 | 1903 |
| 155 | 1903 | 1906 | 1909 | 1912 | 1915 | 1917 | 1920 | 1923 | 1926 | 1928 | 1931 |
| 156 | 1931 | 1934 | 1937 | 1940 | 1942 | 1945 | 1948 | 1951 | 1953 | 1956 | 1959 |
| 157 | 1959 | 1962 | 1965 | 1967 | 1970 | 1973 | 1976 | 1978 | 1981 | 1984 | 1987 |
| 158 | 1987 | 1989 | 1992 | 1995 | 1998 | 2000 | 2003 | 2006 | 2009 | 2011 | 2014 |
| 159 | 2014 | 2017 | 2019 | 2022 | 2025 | 2028 | 2030 | 2033 | 2036 | 2038 | 2041 |
| 160 | 2041 | 2044 | 2047 | 2049 | 2052 | 2055 | 2057 | 2060 | 2063 | 2066 | 2068 |
| 161 | 2068 | 2071 | 2074 | 2076 | 2079 | 2082 | 2084 | 2087 | 2090 | 2092 | 2095 |
| 162 | 2095 | 2098 | 2101 | 2103 | 2106 | 2109 | 2111 | 2114 | 2117 | 2119 | 2122 |
| 163 | 2122 | 2125 | 2127 | 2130 | 2133 | 2135 | 2138 | 2140 | 2143 | 2146 | 2148 |
| 164 | 2148 | 2151 | 2154 | 2156 | 2159 | 2162 | 2164 | 2167 | 2170 | 2172 | 2175 |
| 165 | 2175 | 2177 | 2180 | 2183 | 2185 | 2188 | 2191 | 2193 | 2196 | 2198 | 2201 |
| 166 | 2201 | 2204 | 2206 | 2209 | 2212 | 2214 | 2217 | 2219 | 2222 | 2225 | 2227 |
| 167 | 2227 | 2230 | 2232 | 2235 | 2238 | 2240 | 2243 | 2245 | 2248 | 2251 | 2253 |
| 168 | 2253 | 2256 | 2258 | 2261 | 2263 | 2266 | 2269 | 2271 | 2274 | 2276 | 2279 |
| 169 | 2279 | 2281 | 2284 | 2287 | 2289 | 2292 | 2294 | 2297 | 2299 | 2302 | 2304 |
| 170 | 2304 | 2307 | 2310 | 2312 | 2315 | 2317 | 2320 | 2322 | 2325 | 2327 | 2330 |
| 171 | 2330 | 2333 | 2335 | 2338 | 2340 | 2343 | 2345 | 2348 | 2350 | 2353 | 2355 |
| 172 | 2355 | 2358 | 2360 | 2363 | 2365 | 2368 | 2370 | 2373 | 2375 | 2378 | 2380 |
| 173 | 2380 | 2383 | 2385 | 2388 | 2390 | 2393 | 2395 | 2398 | 2400 | 2403 | 2405 |
| 174 | 2405 | 2408 | 2410 | 2413 | 2415 | 2418 | 2420 | 2423 | 2425 | 2428 | 2430 |
| 175 | 2430 | 2433 | 2435 | 2438 | 2440 | 2443 | 2445 | 2448 | 2450 | 2453 | 2455 |
| 176 | 2455 | 2458 | 2460 | 2463 | 2465 | 2467 | 2470 | 2472 | 2475 | 2477 | 2480 |
| 177 | 2480 | 2482 | 2485 | 2487 | 2490 | 2492 | 2494 | 2497 | 2499 | 2502 | 2504 |
| 178 | 2504 | 2507 | 2509 | 2512 | 2514 | 2516 | 2519 | 2521 | 2524 | 2526 | 2529 |
| 179 | 2529 | 2531 | 2533 | 2536 | 2538 | 2541 | 2543 | 2545 | 2548 | 2550 | 2553 |
| 180 | 2553 | 2555 | 2558 | 2560 | 2562 | 2565 | 2567 | 2570 | 2572 | 2574 | 2577 |
| 181 | 2577 | 2579 | 2582 | 2584 | 2586 | 2589 | 2591 | 2594 | 2596 | 2598 | 2601 |
| 182 | 2601 | 2603 | 2605 | 2608 | 2610 | 2613 | 2615 | 2617 | 2620 | 2622 | 2625 |
| 183 | 2625 | 2627 | 2629 | 2632 | 2634 | 2636 | 2639 | 2641 | 2643 | 2646 | 2648 |
| 184 | 2648 | 2651 | 2653 | 2655 | 2658 | 2660 | 2662 | 2665 | 2667 | 2669 | 2672 |
| 185 | 2672 | 2674 | 2676 | 2679 | 2681 | 2683 | 2686 | 2688 | 2690 | 2693 | 2695 |
| 186 | 2695 | 2697 | 2700 | 2702 | 2704 | 2707 | 2709 | 2711 | 2714 | 2716 | 2718 |
| 187 | 2718 | 2721 | 2723 | 2725 | 2728 | 2730 | 2732 | 2735 | 2737 | 2739 | 2742 |
| 188 | 2742 | 2744 | 2746 | 2749 | 2751 | 2753 | 2755 | 2758 | 2760 | 2762 | 2765 |
| 189 | 2765 | 2767 | 2769 | 2772 | 2774 | 2776 | 2778 | 2781 | 2783 | 2785 | 2788 |
| 190 | 2788 | 2790 | 2792 | 2794 | 2797 | 2799 | 2801 | 2804 | 2806 | 2808 | 2810 |
| 191 | 2810 | 2813 | 2815 | 2817 | 2819 | 2822 | 2824 | 2826 | 2828 | 2831 | 2833 |
| 192 | 2833 | 2835 | 2838 | 2840 | 2842 | 2844 | 2847 | 2849 | 2851 | 2853 | 2856 |
| 193 | 2856 | 2858 | 2860 | 2862 | 2865 | 2867 | 2869 | 2871 | 2874 | 2876 | 2878 |
| 194 | 2878 | 2880 | 2882 | 2885 | 2887 | 2889 | 2891 | 2894 | 2896 | 2898 | 2900 |
| 195 | 2900 | 2903 | 2905 | 2907 | 2909 | 2911 | 2914 | 2916 | 2918 | 2920 | 2923 |
| 196 | 2923 | 2925 | 2927 | 2929 | 2931 | 2934 | 2936 | 2938 | 2940 | 2942 | 2945 |
| 197 | 2945 | 2947 | 2949 | 2951 | 2953 | 2956 | 2958 | 2960 | 2962 | 2964 | 2967 |
| 198 | 2967 | 2969 | 2971 | 2973 | 2975 | 2978 | 2980 | 2982 | 2984 | 2986 | 2989 |
| 199 | 2989 | 2991 | 2993 | 2995 | 2997 | 2999 | 3002 | 3004 | 3006 | 3008 | 3010 |

Trigonometric Functions.

| RADIAN. | DEGREES. | SINES. | | COSINES. | | TANGENTS. | | COTANGENTS. | | | |
|---------|----------|----------|---------------|----------|---------------|-------------|---------------|---------------|---------------|----------|----------|
| | | Nat. | Log. ∞ | Nat. | Log. ∞ | Nat. | Log. ∞ | Nat. ∞ | Log. ∞ | | |
| 0.0000 | 0° 00' | .0000 | | 1.0000 | 0.0000 | .0000 | | | | 90° 00' | 1.5708 |
| 0.0029 | 10 | .0029 | 7.4637 | 1.0000 | .0000 | .0029 | 7.4637 | 343.77 | 2.5363 | 50 | 1.5679 |
| 0.0058 | 20 | .0058 | .7648 | 1.0000 | .0000 | .0058 | .7648 | 171.89 | .2352 | 40 | 1.5650 |
| 0.0087 | 30 | .0087 | .9408 | 1.0000 | .0000 | .0087 | .9409 | 114.59 | .0591 | 30 | 1.5621 |
| 0.0116 | 40 | .0116 | 8.0658 | .9999 | .0000 | .0116 | 8.0658 | 85.940 | 1.9342 | 20 | 1.5592 |
| 0.0145 | 50 | .0145 | .1627 | .9999 | .0000 | .0145 | .1627 | 68.750 | .8373 | 10 | 1.5563 |
| 0.0175 | 1° 00' | .0175 | 8.2419 | .9998 | 9.9999 | .0175 | 8.2419 | 57.290 | 1.7581 | 89° 00' | 1.5533 |
| 0.0204 | 10 | .0204 | .3088 | .9998 | .9999 | .0204 | .3089 | 49.104 | .6911 | 50 | 1.5504 |
| 0.0233 | 20 | .0233 | .3668 | .9997 | .9999 | .0233 | .3669 | 42.964 | .6331 | 40 | 1.5475 |
| 0.0262 | 30 | .0262 | .4179 | .9997 | .9999 | .0262 | .4181 | 38.188 | .5819 | 30 | 1.5446 |
| 0.0291 | 40 | .0291 | .4637 | .9996 | .9998 | .0291 | .4638 | 34.368 | .5362 | 20 | 1.5417 |
| 0.0320 | 50 | .0320 | .5050 | .9995 | .9998 | .0320 | .5053 | 31.242 | .4947 | 10 | 1.5388 |
| 0.0349 | 2° 00' | .0349 | 8.5428 | .9994 | 9.9997 | .0349 | 8.5431 | 28.636 | 1.4569 | 88° 00' | 1.5359 |
| 0.0378 | 10 | .0378 | .5776 | .9993 | .9997 | .0378 | .5779 | 26.432 | .4221 | 50 | 1.5330 |
| 0.0407 | 20 | .0407 | .6097 | .9992 | .9996 | .0407 | .6101 | 24.542 | .3899 | 40 | 1.5301 |
| 0.0436 | 30 | .0436 | .6397 | .9990 | .9996 | .0437 | .6401 | 22.904 | .3599 | 30 | 1.5272 |
| 0.0465 | 40 | .0465 | .6677 | .9989 | .9995 | .0466 | .6682 | 21.470 | .3318 | 20 | 1.5243 |
| 0.0495 | 50 | .0494 | .6940 | .9988 | .9995 | .0495 | .6945 | 20.206 | .3055 | 10 | 1.5213 |
| 0.0524 | 3° 00' | .0523 | 8.7188 | .9986 | 9.9994 | .0524 | 8.7194 | 19.081 | 1.2806 | 87° 00' | 1.5184 |
| 0.0553 | 10 | .0552 | .7423 | .9985 | .9993 | .0553 | .7429 | 18.075 | .2571 | 50 | 1.5155 |
| 0.0582 | 20 | .0581 | .7645 | .9983 | .9993 | .0582 | .7652 | 17.169 | .2348 | 40 | 1.5126 |
| 0.0611 | 30 | .0610 | .7857 | .9981 | .9992 | .0612 | .7865 | 16.350 | .2135 | 30 | 1.5097 |
| 0.0640 | 40 | .0640 | .8059 | .9980 | .9991 | .0641 | .8067 | 15.605 | .1933 | 20 | 1.5068 |
| 0.0669 | 50 | .0669 | .8251 | .9978 | .9990 | .0670 | .8261 | 14.924 | .1739 | 10 | 1.5039 |
| 0.0698 | 4° 00' | .0698 | 8.8436 | .9976 | 9.9989 | .0699 | 8.8446 | 14.301 | 1.1554 | 86° 00' | 1.5010 |
| 0.0727 | 10 | .0727 | .8613 | .9974 | .9989 | .0729 | .8624 | 13.727 | .1376 | 50 | 1.4981 |
| 0.0756 | 20 | .0756 | .8783 | .9971 | .9988 | .0758 | .8795 | 13.197 | .1205 | 40 | 1.4952 |
| 0.0785 | 30 | .0785 | .8946 | .9969 | .9987 | .0787 | .8960 | 12.706 | .1040 | 30 | 1.4923 |
| 0.0814 | 40 | .0814 | .9104 | .9967 | .9986 | .0816 | .9118 | 12.251 | .0882 | 20 | 1.4893 |
| 0.0844 | 50 | .0843 | .9256 | .9964 | .9985 | .0846 | .9272 | 11.826 | .0728 | 10 | 1.4864 |
| 0.0873 | 5° 00' | .0872 | 8.9403 | .9962 | 9.9983 | .0875 | 8.9420 | 11.430 | 1.0580 | 85° 00' | 1.4835 |
| 0.0902 | 10 | .0901 | .9545 | .9959 | .9982 | .0904 | .9563 | 11.059 | .0437 | 50 | 1.4806 |
| 0.0931 | 20 | .0929 | .9682 | .9957 | .9981 | .0934 | .9701 | 10.712 | .0299 | 40 | 1.4777 |
| 0.0960 | 30 | .0958 | .9816 | .9954 | .9980 | .0963 | .9836 | 10.385 | .0164 | 30 | 1.4748 |
| 0.0989 | 40 | .0987 | .9945 | .9951 | .9979 | .0992 | .9966 | 10.078 | .0034 | 20 | 1.4719 |
| 0.1018 | 50 | .1016 | 9.0070 | .9948 | .9977 | .1022 | 9.0093 | 9.7882 | 0.9907 | 10 | 1.4690 |
| 0.1047 | 6° 00' | .1045 | 9.0192 | .9945 | 9.9976 | .1051 | 9.0216 | 9.5144 | 0.9784 | 84° 00' | 1.4661 |
| 0.1076 | 10 | .1074 | .0311 | .9942 | .9975 | .1080 | .0336 | 9.2553 | .9664 | 50 | 1.4632 |
| 0.1105 | 20 | .1103 | .0426 | .9939 | .9973 | .1110 | .0453 | 9.0098 | .9547 | 40 | 1.4603 |
| 0.1134 | 30 | .1132 | .0539 | .9936 | .9972 | .1139 | .0567 | 8.7769 | .9433 | 30 | 1.4574 |
| 0.1164 | 40 | .1161 | .0648 | .9932 | .9971 | .1169 | .0678 | 8.5555 | .9322 | 20 | 1.4544 |
| 0.1193 | 50 | .1190 | .0755 | .9929 | .9969 | .1198 | .0786 | 8.3450 | .9214 | 10 | 1.4515 |
| 0.1222 | 7° 00' | .1219 | 9.0859 | .9925 | 9.9968 | .1228 | 9.0891 | 8.1443 | 0.9109 | 83° 00' | 1.4486 |
| 0.1251 | 10 | .1248 | .0961 | .9922 | .9966 | .1257 | .0995 | 7.9530 | .9005 | 50 | 1.4457 |
| 0.1280 | 20 | .1276 | .1060 | .9918 | .9964 | .1287 | .1096 | 7.7704 | .8904 | 40 | 1.4428 |
| 0.1309 | 30 | .1305 | .1157 | .9914 | .9963 | .1317 | .1194 | 7.5958 | .8806 | 30 | 1.4399 |
| 0.1338 | 40 | .1334 | .1252 | .9911 | .9961 | .1346 | .1291 | 7.4287 | .8709 | 20 | 1.4370 |
| 0.1367 | 50 | .1363 | .1345 | .9907 | .9959 | .1376 | .1385 | 7.2687 | .8615 | 10 | 1.4341 |
| 0.1396 | 8° 00' | .1392 | 9.1436 | .9903 | 9.9958 | .1405 | 9.1478 | 7.1154 | 0.8522 | 82° 00' | 1.4312 |
| 0.1425 | 10 | .1421 | .1525 | .9899 | .9956 | .1435 | .1569 | 6.9682 | .8431 | 50 | 1.4283 |
| 0.1454 | 20 | .1449 | .1612 | .9894 | .9954 | .1465 | .1658 | 6.8269 | .8342 | 40 | 1.4254 |
| 0.1484 | 30 | .1478 | .1697 | .9890 | .9952 | .1495 | .1745 | 6.6912 | .8255 | 30 | 1.4224 |
| 0.1513 | 40 | .1507 | .1781 | .9886 | .9950 | .1524 | .1831 | 6.5606 | .8169 | 20 | 1.4195 |
| 0.1542 | 50 | .1536 | .1863 | .9881 | .9948 | .1554 | .1915 | 6.4348 | .8085 | 10 | 1.4166 |
| 0.1571 | 9° 00' | .1564 | 9.1943 | .9877 | 9.9946 | .1584 | 9.1997 | 6.3138 | 0.8003 | 81° 00' | 1.4137 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

Trigonometric Functions.

| RADIANS. | DEGREES. | SINES. | | COSINES. | | TANGENTS. | | COTANGENTS. | | | |
|----------|----------|----------|--------|----------|--------|-------------|--------|-------------|--------|----------|----------|
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| 0.1571 | 9° 00' | .1564 | 9.1943 | .9877 | 9.9946 | .1584 | 9.1997 | 6.3138 | 0.8003 | 81° 00' | 1.4137 |
| 0.1600 | 10 | .1593 | .2022 | .9872 | .9944 | .1614 | .2078 | 6.1970 | .7922 | 50 | 1.4108 |
| 0.1629 | 20 | .1622 | .2100 | .9868 | .9940 | .1644 | .2158 | 6.0844 | .7842 | 40 | 1.4079 |
| 0.1658 | 30 | .1650 | .2176 | .9863 | .9942 | .1673 | .2236 | 5.9758 | .7764 | 30 | 1.4050 |
| 0.1687 | 40 | .1679 | .2251 | .9858 | .9938 | .1703 | .2313 | 5.8708 | .7687 | 20 | 1.4021 |
| 0.1716 | 50 | .1708 | .2324 | .9853 | .9936 | .1733 | .2389 | 5.7694 | .7611 | 10 | 1.3992 |
| 0.1745 | 10° 00' | .1736 | 9.2397 | .9848 | 9.9934 | .1763 | 9.2463 | 5.6713 | 0.7537 | 80° 00' | 1.3963 |
| 0.1774 | 10 | .1765 | .2468 | .9843 | .9931 | .1793 | .2536 | 5.5764 | .7464 | 50 | 1.3934 |
| 0.1804 | 20 | .1794 | .2538 | .9838 | .9929 | .1823 | .2609 | 5.4845 | .7391 | 40 | 1.3904 |
| 0.1833 | 30 | .1822 | .2606 | .9833 | .9927 | .1853 | .2680 | 5.3955 | .7320 | 30 | 1.3875 |
| 0.1862 | 40 | .1851 | .2674 | .9827 | .9924 | .1883 | .2750 | 5.3093 | .7250 | 20 | 1.3846 |
| 0.1891 | 50 | .1880 | .2740 | .9822 | .9922 | .1914 | .2819 | 5.2257 | .7181 | 10 | 1.3817 |
| 0.1920 | 11° 00' | .1908 | 9.2806 | .9816 | 9.9919 | .1944 | 9.2887 | 5.1446 | 0.7113 | 79° 00' | 1.3788 |
| 0.1949 | 10 | .1937 | .2870 | .9811 | .9917 | .1974 | .2953 | 5.0658 | .7047 | 50 | 1.3759 |
| 0.1978 | 20 | .1965 | .2934 | .9805 | .9914 | .2004 | .3020 | 4.9894 | .6980 | 40 | 1.3730 |
| 0.2007 | 30 | .1994 | .2997 | .9799 | .9912 | .2035 | .3085 | 4.9152 | .6915 | 30 | 1.3701 |
| 0.2036 | 40 | .2022 | .3058 | .9793 | .9909 | .2065 | .3149 | 4.8430 | .6851 | 20 | 1.3672 |
| 0.2065 | 50 | .2051 | .3119 | .9787 | .9907 | .2095 | .3212 | 4.7729 | .6788 | 10 | 1.3643 |
| 0.2094 | 12° 00' | .2079 | 9.3179 | .9781 | 9.9904 | .2126 | 9.3275 | 4.7046 | 0.6725 | 78° 00' | 1.3614 |
| 0.2123 | 10 | .2108 | .3238 | .9775 | .9901 | .2156 | .3336 | 4.6382 | .6664 | 50 | 1.3584 |
| 0.2153 | 20 | .2136 | .3296 | .9769 | .9899 | .2186 | .3397 | 4.5736 | .6603 | 40 | 1.3555 |
| 0.2182 | 30 | .2164 | .3353 | .9763 | .9896 | .2217 | .3458 | 4.5107 | .6542 | 30 | 1.3526 |
| 0.2211 | 40 | .2193 | .3410 | .9757 | .9893 | .2247 | .3517 | 4.4494 | .6483 | 20 | 1.3497 |
| 0.2240 | 50 | .2221 | .3466 | .9750 | .9890 | .2278 | .3576 | 4.3897 | .6424 | 10 | 1.3468 |
| 0.2269 | 13° 00' | .2250 | 9.3521 | .9744 | 9.9887 | .2309 | 9.3634 | 4.3315 | 0.6366 | 77° 00' | 1.3439 |
| 0.2298 | 10 | .2278 | .3575 | .9737 | .9884 | .2339 | .3691 | 4.2747 | .6309 | 50 | 1.3410 |
| 0.2327 | 20 | .2306 | .3629 | .9730 | .9881 | .2370 | .3748 | 4.2193 | .6252 | 40 | 1.3381 |
| 0.2356 | 30 | .2334 | .3682 | .9724 | .9878 | .2401 | .3804 | 4.1653 | .6196 | 30 | 1.3352 |
| 0.2385 | 40 | .2363 | .3734 | .9717 | .9875 | .2432 | .3859 | 4.1126 | .6141 | 20 | 1.3323 |
| 0.2414 | 50 | .2391 | .3786 | .9710 | .9872 | .2462 | .3914 | 4.0611 | .6086 | 10 | 1.3294 |
| 0.2443 | 14° 00' | .2419 | 9.3837 | .9703 | 9.9869 | .2493 | 9.3968 | 4.0108 | 0.6032 | 76° 00' | 1.3265 |
| 0.2473 | 10 | .2447 | .3887 | .9696 | .9866 | .2524 | .4021 | 3.9617 | .5979 | 50 | 1.3235 |
| 0.2502 | 20 | .2476 | .3937 | .9689 | .9863 | .2555 | .4074 | 3.9136 | .5926 | 40 | 1.3206 |
| 0.2531 | 30 | .2504 | .3986 | .9681 | .9859 | .2586 | .4127 | 3.8667 | .5873 | 30 | 1.3177 |
| 0.2560 | 40 | .2532 | .4035 | .9674 | .9856 | .2617 | .4178 | 3.8208 | .5822 | 20 | 1.3148 |
| 0.2589 | 50 | .2560 | .4083 | .9667 | .9853 | .2648 | .4230 | 3.7760 | .5770 | 10 | 1.3119 |
| 0.2618 | 15° 00' | .2588 | 9.4130 | .9659 | 9.9849 | .2679 | 9.4281 | 3.7321 | 0.5719 | 75° 00' | 1.3090 |
| 0.2647 | 10 | .2616 | .4177 | .9652 | .9846 | .2711 | .4331 | 3.6891 | .5669 | 50 | 1.3061 |
| 0.2676 | 20 | .2644 | .4223 | .9644 | .9843 | .2742 | .4381 | 3.6470 | .5619 | 40 | 1.3032 |
| 0.2705 | 30 | .2672 | .4269 | .9636 | .9839 | .2773 | .4430 | 3.6059 | .5570 | 30 | 1.3003 |
| 0.2734 | 40 | .2700 | .4314 | .9628 | .9836 | .2805 | .4479 | 3.5656 | .5521 | 20 | 1.2974 |
| 0.2763 | 50 | .2728 | .4359 | .9621 | .9832 | .2836 | .4527 | 3.5261 | .5473 | 10 | 1.2945 |
| 0.2793 | 16° 00' | .2756 | 9.4403 | .9613 | 9.9828 | .2867 | 9.4575 | 3.4874 | 0.5425 | 74° 00' | 1.2915 |
| 0.2822 | 10 | .2784 | .4447 | .9605 | .9825 | .2899 | .4622 | 3.4495 | .5378 | 50 | 1.2886 |
| 0.2851 | 20 | .2812 | .4491 | .9596 | .9821 | .2931 | .4669 | 3.4124 | .5331 | 40 | 1.2857 |
| 0.2880 | 30 | .2840 | .4533 | .9588 | .9817 | .2962 | .4716 | 3.3759 | .5284 | 30 | 1.2828 |
| 0.2909 | 40 | .2868 | .4576 | .9580 | .9814 | .2994 | .4762 | 3.3402 | .5238 | 20 | 1.2799 |
| 0.2938 | 50 | .2896 | .4618 | .9572 | .9810 | .3026 | .4808 | 3.3052 | .5192 | 10 | 1.2770 |
| 0.2967 | 17° 00' | .2924 | 9.4659 | .9563 | 9.9806 | .3057 | 9.4853 | 3.2709 | 0.5147 | 73° 00' | 1.2741 |
| 0.2996 | 10 | .2952 | .4700 | .9555 | .9802 | .3089 | .4898 | 3.2371 | .5102 | 50 | 1.2712 |
| 0.3025 | 20 | .2979 | .4741 | .9546 | .9798 | .3121 | .4943 | 3.2041 | .5057 | 40 | 1.2683 |
| 0.3054 | 30 | .3007 | .4781 | .9537 | .9794 | .3153 | .4987 | 3.1716 | .5013 | 30 | 1.2654 |
| 0.3083 | 40 | .3035 | .4821 | .9528 | .9790 | .3185 | .5031 | 3.1397 | .4969 | 20 | 1.2625 |
| 0.3113 | 50 | .3062 | .4861 | .9520 | .9786 | .3217 | .5075 | 3.1084 | .4925 | 10 | 1.2595 |
| 0.3142 | 18° 00' | .3090 | 9.4900 | .9511 | 9.9782 | .3249 | 9.5118 | 3.0777 | 0.4882 | 72° 00' | 1.2566 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

Trigonometric Functions.

| RADIANs. | DEGREES. | SINES. | | COSINES. | | TANGENTS. | | COTANGENTS. | | | |
|----------|----------|----------|--------|----------|--------|-------------|--------|-------------|--------|----------|----------|
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| 0.3142 | 18° 00' | .3090 | 9.4900 | .9511 | 9.9782 | .3249 | 9.5118 | 3.0777 | 0.4882 | 72° 00' | 1.2566 |
| 0.3171 | 10 | .3118 | .4939 | .9502 | .9778 | .3281 | .5161 | 3.0475 | .4839 | 50 | 1.2537 |
| 0.3200 | 20 | .3145 | .4977 | .9492 | .9774 | .3314 | .5203 | 3.0178 | .4797 | 40 | 1.2508 |
| 0.3229 | 30 | .3173 | .5015 | .9483 | .9770 | .3346 | .5245 | 2.9887 | .4755 | 30 | 1.2479 |
| 0.3258 | 40 | .3201 | .5052 | .9474 | .9765 | .3378 | .5287 | 2.9600 | .4713 | 20 | 1.2450 |
| 0.3287 | 50 | .3228 | .5090 | .9465 | .9761 | .3411 | .5329 | 2.9319 | .4671 | 10 | 1.2421 |
| 0.3316 | 19° 00' | .3256 | 9.5126 | .9455 | 9.9757 | .3443 | 9.5370 | 2.9042 | 0.4630 | 71° 00' | 1.2392 |
| 0.3345 | 10 | .3283 | .5163 | .9446 | .9752 | .3476 | .5411 | 2.8770 | .4589 | 50 | 1.2363 |
| 0.3374 | 20 | .3311 | .5199 | .9436 | .9748 | .3508 | .5451 | 2.8502 | .4549 | 40 | 1.2334 |
| 0.3403 | 30 | .3338 | .5235 | .9426 | .9743 | .3541 | .5491 | 2.8239 | .4509 | 30 | 1.2305 |
| 0.3432 | 40 | .3365 | .5270 | .9417 | .9739 | .3574 | .5531 | 2.7980 | .4469 | 20 | 1.2275 |
| 0.3462 | 50 | .3393 | .5306 | .9407 | .9734 | .3607 | .5571 | 2.7725 | .4429 | 10 | 1.2246 |
| 0.3491 | 20° 00' | .3420 | 9.5341 | .9397 | 9.9730 | .3640 | 9.5611 | 2.7475 | 0.4389 | 70° 00' | 1.2217 |
| 0.3520 | 10 | .3448 | .5375 | .9387 | .9725 | .3673 | .5650 | 2.7228 | .4350 | 50 | 1.2188 |
| 0.3549 | 20 | .3475 | .5409 | .9377 | .9721 | .3706 | .5689 | 2.6985 | .4311 | 40 | 1.2159 |
| 0.3578 | 30 | .3502 | .5443 | .9367 | .9716 | .3739 | .5727 | 2.6746 | .4273 | 30 | 1.2130 |
| 0.3607 | 40 | .3529 | .5477 | .9356 | .9711 | .3772 | .5766 | 2.6511 | .4234 | 20 | 1.2101 |
| 0.3636 | 50 | .3557 | .5510 | .9346 | .9706 | .3805 | .5804 | 2.6279 | .4196 | 10 | 1.2072 |
| 0.3665 | 21° 00' | .3584 | 9.5543 | .9336 | 9.9702 | .3839 | 9.5842 | 2.6051 | 0.4158 | 69° 00' | 1.2043 |
| 0.3694 | 10 | .3611 | .5576 | .9325 | .9697 | .3872 | .5879 | 2.5826 | .4121 | 50 | 1.2014 |
| 0.3723 | 20 | .3638 | .5609 | .9315 | .9692 | .3906 | .5917 | 2.5605 | .4083 | 40 | 1.1985 |
| 0.3752 | 30 | .3665 | .5641 | .9304 | .9687 | .3939 | .5954 | 2.5386 | .4046 | 30 | 1.1956 |
| 0.3782 | 40 | .3692 | .5673 | .9293 | .9682 | .3973 | .5991 | 2.5172 | .4009 | 20 | 1.1926 |
| 0.3811 | 50 | .3719 | .5704 | .9283 | .9677 | .4006 | .6028 | 2.4960 | .3972 | 10 | 1.1897 |
| 0.3840 | 22° 00' | .3746 | 9.5736 | .9272 | 9.9672 | .4040 | 9.6064 | 2.4751 | 0.3936 | 68° 00' | 1.1868 |
| 0.3869 | 10 | .3773 | .5767 | .9261 | .9667 | .4074 | .6100 | 2.4545 | .3900 | 50 | 1.1839 |
| 0.3898 | 20 | .3800 | .5798 | .9250 | .9661 | .4108 | .6136 | 2.4342 | .3864 | 40 | 1.1810 |
| 0.3927 | 30 | .3827 | .5828 | .9239 | .9656 | .4142 | .6172 | 2.4142 | .3828 | 30 | 1.1781 |
| 0.3956 | 40 | .3854 | .5859 | .9228 | .9651 | .4176 | .6208 | 2.3945 | .3792 | 20 | 1.1752 |
| 0.3985 | 50 | .3881 | .5889 | .9216 | .9646 | .4210 | .6243 | 2.3750 | .3757 | 10 | 1.1723 |
| 0.4014 | 23° 00' | .3907 | 9.5919 | .9205 | 9.9640 | .4245 | 9.6279 | 2.3559 | 0.3721 | 67° 00' | 1.1694 |
| 0.4043 | 10 | .3934 | .5948 | .9194 | .9635 | .4279 | .6314 | 2.3369 | .3686 | 50 | 1.1665 |
| 0.4072 | 20 | .3961 | .5978 | .9182 | .9629 | .4314 | .6348 | 2.3183 | .3652 | 40 | 1.1636 |
| 0.4102 | 30 | .3987 | .6007 | .9171 | .9624 | .4348 | .6383 | 2.2998 | .3617 | 30 | 1.1606 |
| 0.4131 | 40 | .4014 | .6036 | .9159 | .9618 | .4383 | .6417 | 2.2817 | .3583 | 20 | 1.1577 |
| 0.4160 | 50 | .4041 | .6065 | .9147 | .9613 | .4417 | .6452 | 2.2637 | .3548 | 10 | 1.1548 |
| 0.4189 | 24° 00' | .4067 | 9.6093 | .9135 | 9.9607 | .4452 | 9.6486 | 2.2460 | 0.3514 | 66° 00' | 1.1519 |
| 0.4218 | 10 | .4094 | .6121 | .9124 | .9602 | .4487 | .6520 | 2.2286 | .3480 | 50 | 1.1490 |
| 0.4247 | 20 | .4120 | .6149 | .9112 | .9596 | .4522 | .6553 | 2.2113 | .3447 | 40 | 1.1461 |
| 0.4276 | 30 | .4147 | .6177 | .9100 | .9590 | .4557 | .6587 | 2.1943 | .3413 | 30 | 1.1432 |
| 0.4305 | 40 | .4173 | .6205 | .9088 | .9584 | .4592 | .6620 | 2.1775 | .3380 | 20 | 1.1403 |
| 0.4334 | 50 | .4200 | .6232 | .9075 | .9579 | .4628 | .6654 | 2.1609 | .3346 | 10 | 1.1374 |
| 0.4363 | 25° 00' | .4226 | 9.6259 | .9063 | 9.9573 | .4663 | 9.6687 | 2.1445 | 0.3313 | 65° 00' | 1.1345 |
| 0.4392 | 10 | .4253 | .6286 | .9051 | .9567 | .4699 | .6720 | 2.1283 | .3280 | 50 | 1.1316 |
| 0.4422 | 20 | .4279 | .6313 | .9038 | .9561 | .4734 | .6752 | 2.1123 | .3248 | 40 | 1.1286 |
| 0.4451 | 30 | .4305 | .6340 | .9026 | .9555 | .4770 | .6785 | 2.0965 | .3215 | 30 | 1.1257 |
| 0.4480 | 40 | .4331 | .6366 | .9013 | .9549 | .4806 | .6817 | 2.0809 | .3183 | 20 | 1.1228 |
| 0.4509 | 50 | .4358 | .6392 | .9001 | .9543 | .4841 | .6850 | 2.0655 | .3150 | 10 | 1.1199 |
| 0.4538 | 26° 00' | .4384 | 9.6418 | .8988 | 9.9537 | .4877 | 9.6882 | 2.0503 | 0.3118 | 64° 00' | 1.1170 |
| 0.4567 | 10 | .4410 | .6444 | .8975 | .9530 | .4913 | .6914 | 2.0353 | .3086 | 50 | 1.1141 |
| 0.4596 | 20 | .4436 | .6470 | .8962 | .9524 | .4950 | .6946 | 2.0204 | .3054 | 40 | 1.1112 |
| 0.4625 | 30 | .4462 | .6495 | .8949 | .9518 | .4986 | .6977 | 2.0057 | .3023 | 30 | 1.1083 |
| 0.4654 | 40 | .4488 | .6521 | .8936 | .9512 | .5022 | .7009 | 1.9912 | .2991 | 20 | 1.1054 |
| 0.4683 | 50 | .4514 | .6546 | .8923 | .9505 | .5059 | .7040 | 1.9768 | .2960 | 10 | 1.1025 |
| 0.4712 | 27° 00' | .4540 | 9.6570 | .8910 | 9.9499 | .5095 | 9.7072 | 1.9626 | 0.2928 | 63° 00' | 1.0996 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

Trigonometric Functions.

| RADIAN. | DEGREES. | SINES. | | COSINES. | | TANGENTS. | | COTANGENTS. | | | |
|---------|----------|----------|--------|----------|--------|-------------|--------|-------------|--------|----------|----------|
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| 0.4712 | 27° 00' | .4540 | 9.6570 | .8910 | 9.9499 | .5095 | 9.7072 | 1.9626 | 0.2928 | 63° 00' | 1.0996 |
| 0.4741 | 10 | .4566 | .6595 | .8897 | .9492 | .5132 | .7103 | 1.9486 | .2897 | 50 | 1.0966 |
| 0.4771 | 20 | .4592 | .6620 | .8884 | .9486 | .5169 | .7134 | 1.9347 | .2866 | 40 | 1.0937 |
| 0.4800 | 30 | .4617 | .6644 | .8870 | .9479 | .5206 | .7165 | 1.9210 | .2835 | 30 | 1.0908 |
| 0.4829 | 40 | .4643 | .6668 | .8857 | .9473 | .5243 | .7196 | 1.9074 | .2804 | 20 | 1.0879 |
| 0.4858 | 50 | .4669 | .6692 | .8843 | .9466 | .5280 | .7226 | 1.8940 | .2774 | 10 | 1.0850 |
| 0.4887 | 28° 00' | .4695 | 9.6716 | .8829 | 9.9459 | .5317 | 9.7257 | 1.8807 | 0.2743 | 62° 00' | 1.0821 |
| 0.4916 | 10 | .4720 | .6740 | .8816 | .9453 | .5354 | .7287 | 1.8676 | .2713 | 50 | 1.0792 |
| 0.4945 | 20 | .4746 | .6763 | .8802 | .9446 | .5392 | .7317 | 1.8546 | .2683 | 40 | 1.0763 |
| 0.4974 | 30 | .4772 | .6787 | .8788 | .9439 | .5430 | .7348 | 1.8418 | .2652 | 30 | 1.0734 |
| 0.5003 | 40 | .4797 | .6810 | .8774 | .9432 | .5467 | .7378 | 1.8291 | .2622 | 20 | 1.0705 |
| 0.5032 | 50 | .4823 | .6833 | .8760 | .9425 | .5505 | .7408 | 1.8165 | .2592 | 10 | 1.0676 |
| 0.5061 | 29° 00' | .4848 | 9.6856 | .8746 | 9.9418 | .5543 | 9.7438 | 1.8040 | 0.2562 | 61° 00' | 1.0647 |
| 0.5091 | 10 | .4874 | .6878 | .8732 | .9411 | .5581 | .7467 | 1.7917 | .2533 | 50 | 1.0617 |
| 0.5120 | 20 | .4899 | .6901 | .8718 | .9404 | .5619 | .7497 | 1.7796 | .2503 | 40 | 1.0588 |
| 0.5149 | 30 | .4924 | .6923 | .8704 | .9397 | .5658 | .7526 | 1.7675 | .2474 | 30 | 1.0559 |
| 0.5178 | 40 | .4950 | .6946 | .8689 | .9390 | .5696 | .7556 | 1.7556 | .2444 | 20 | 1.0530 |
| 0.5207 | 50 | .4975 | .6968 | .8675 | .9383 | .5735 | .7585 | 1.7437 | .2415 | 10 | 1.0501 |
| 0.5236 | 30° 00' | .5000 | 9.6990 | .8660 | 9.9375 | .5774 | 9.7614 | 1.7321 | 0.2386 | 60° 00' | 1.0472 |
| 0.5265 | 10 | .5025 | .7012 | .8646 | .9368 | .5812 | .7644 | 1.7205 | .2356 | 50 | 1.0443 |
| 0.5294 | 20 | .5050 | .7033 | .8631 | .9361 | .5851 | .7673 | 1.7090 | .2327 | 40 | 1.0414 |
| 0.5323 | 30 | .5075 | .7055 | .8616 | .9353 | .5890 | .7701 | 1.6977 | .2299 | 30 | 1.0385 |
| 0.5352 | 40 | .5100 | .7076 | .8601 | .9346 | .5930 | .7730 | 1.6864 | .2270 | 20 | 1.0356 |
| 0.5381 | 50 | .5125 | .7097 | .8587 | .9338 | .5969 | .7759 | 1.6753 | .2241 | 10 | 1.0327 |
| 0.5411 | 31° 00' | .5150 | 9.7118 | .8572 | 9.9331 | .6009 | 9.7788 | 1.6643 | 0.2212 | 59° 00' | 1.0297 |
| 0.5440 | 10 | .5175 | .7139 | .8557 | .9323 | .6048 | .7816 | 1.6534 | .2184 | 50 | 1.0268 |
| 0.5469 | 20 | .5200 | .7160 | .8542 | .9315 | .6088 | .7845 | 1.6426 | .2155 | 40 | 1.0239 |
| 0.5498 | 30 | .5225 | .7181 | .8526 | .9308 | .6128 | .7873 | 1.6319 | .2127 | 30 | 1.0210 |
| 0.5527 | 40 | .5250 | .7201 | .8511 | .9300 | .6168 | .7902 | 1.6212 | .2098 | 20 | 1.0181 |
| 0.5556 | 50 | .5275 | .7222 | .8496 | .9292 | .6208 | .7930 | 1.6107 | .2070 | 10 | 1.0152 |
| 0.5585 | 32° 00' | .5299 | 9.7242 | .8480 | 9.9284 | .6249 | 9.7958 | 1.6003 | 0.2042 | 58° 00' | 1.0123 |
| 0.5614 | 10 | .5324 | .7262 | .8465 | .9276 | .6289 | .7986 | 1.5900 | .2014 | 50 | 1.0094 |
| 0.5643 | 20 | .5348 | .7282 | .8450 | .9268 | .6330 | .8014 | 1.5798 | .1986 | 40 | 1.0065 |
| 0.5672 | 30 | .5373 | .7302 | .8434 | .9260 | .6371 | .8042 | 1.5697 | .1958 | 30 | 1.0036 |
| 0.5701 | 40 | .5398 | .7322 | .8418 | .9252 | .6412 | .8070 | 1.5597 | .1930 | 20 | 1.0007 |
| 0.5730 | 50 | .5422 | .7342 | .8403 | .9244 | .6453 | .8097 | 1.5497 | .1903 | 10 | 0.9977 |
| 0.5760 | 33° 00' | .5446 | 9.7361 | .8387 | 9.9236 | .6494 | 9.8125 | 1.5399 | 0.1875 | 57° 00' | 0.9948 |
| 0.5789 | 10 | .5471 | .7380 | .8371 | .9228 | .6536 | .8153 | 1.5301 | .1847 | 50 | 0.9919 |
| 0.5818 | 20 | .5495 | .7400 | .8355 | .9219 | .6577 | .8180 | 1.5204 | .1820 | 40 | 0.9890 |
| 0.5847 | 30 | .5519 | .7419 | .8339 | .9211 | .6619 | .8208 | 1.5108 | .1792 | 30 | 0.9861 |
| 0.5876 | 40 | .5544 | .7438 | .8323 | .9203 | .6661 | .8235 | 1.5013 | .1765 | 20 | 0.9832 |
| 0.5905 | 50 | .5568 | .7457 | .8307 | .9194 | .6703 | .8263 | 1.4919 | .1737 | 10 | 0.9803 |
| 0.5934 | 34° 00' | .5592 | 9.7476 | .8290 | 9.9186 | .6745 | 9.8290 | 1.4826 | 0.1710 | 56° 00' | 0.9774 |
| 0.5963 | 10 | .5616 | .7494 | .8274 | .9177 | .6787 | .8317 | 1.4733 | .1683 | 50 | 0.9745 |
| 0.5992 | 20 | .5640 | .7513 | .8258 | .9169 | .6830 | .8344 | 1.4641 | .1656 | 40 | 0.9716 |
| 0.6021 | 30 | .5664 | .7531 | .8241 | .9160 | .6873 | .8371 | 1.4550 | .1629 | 30 | 0.9687 |
| 0.6050 | 40 | .5688 | .7550 | .8225 | .9151 | .6916 | .8398 | 1.4460 | .1602 | 20 | 0.9657 |
| 0.6080 | 50 | .5712 | .7568 | .8208 | .9142 | .6959 | .8425 | 1.4370 | .1575 | 10 | 0.9628 |
| 0.6109 | 35° 00' | .5736 | 9.7586 | .8192 | 9.9134 | .7002 | 9.8452 | 1.4281 | 0.1548 | 55° 00' | 0.9599 |
| 0.6138 | 10 | .5760 | .7604 | .8175 | .9125 | .7046 | .8479 | 1.4193 | .1521 | 50 | 0.9570 |
| 0.6167 | 20 | .5783 | .7622 | .8158 | .9116 | .7089 | .8506 | 1.4106 | .1494 | 40 | 0.9541 |
| 0.6196 | 30 | .5807 | .7640 | .8141 | .9107 | .7133 | .8533 | 1.4019 | .1467 | 30 | 0.9512 |
| 0.6225 | 40 | .5831 | .7657 | .8124 | .9098 | .7177 | .8559 | 1.3934 | .1441 | 20 | 0.9483 |
| 0.6254 | 50 | .5854 | .7675 | .8107 | .9089 | .7221 | .8586 | 1.3848 | .1414 | 10 | 0.9454 |
| 0.6283 | 36° 00' | .5878 | 9.7692 | .8090 | 9.9080 | .7265 | 9.8613 | 1.3764 | 0.1387 | 54° 00' | 0.9425 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

Trigonometric Functions.

| RADIAN. | DEGREES. | SINES. | | COSINES. | | TANGENTS. | | COTANGENTS. | | | |
|---------|----------|----------|--------|----------|--------|-------------|--------|-------------|--------|----------|----------|
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| 0.6283 | 36° 00' | .5878 | 9.7692 | .8090 | 9.9080 | .7265 | 9.8613 | 1.3764 | 0.1387 | 54° 00' | 0.9425 |
| 0.6312 | 10 | .5901 | .7710 | .8073 | .9070 | .7310 | .8639 | 1.3680 | .1361 | 50 | 0.9396 |
| 0.6341 | 20 | .5925 | .7727 | .8056 | .9061 | .7355 | .8666 | 1.3597 | .1334 | 40 | 0.9367 |
| 0.6370 | 30 | .5948 | .7744 | .8039 | .9052 | .7400 | .8692 | 1.3514 | .1308 | 30 | 0.9338 |
| 0.6400 | 40 | .5972 | .7761 | .8021 | .9042 | .7445 | .8718 | 1.3432 | .1282 | 20 | 0.9308 |
| 0.6429 | 50 | .5995 | .7778 | .8004 | .9033 | .7490 | .8745 | 1.3351 | .1255 | 10 | 0.9279 |
| 0.6458 | 37° 00' | .6018 | 9.7795 | .7986 | 9.9023 | .7536 | 9.8771 | 1.3270 | 0.1229 | 53° 00' | 0.9250 |
| 0.6487 | 10 | .6041 | .7811 | .7969 | .9014 | .7581 | .8797 | 1.3190 | .1203 | 50 | 0.9221 |
| 0.6516 | 20 | .6065 | .7828 | .7951 | .9004 | .7627 | .8824 | 1.3111 | .1176 | 40 | 0.9192 |
| 0.6545 | 30 | .6088 | .7844 | .7934 | .8995 | .7673 | .8850 | 1.3032 | .1150 | 30 | 0.9163 |
| 0.6574 | 40 | .6111 | .7861 | .7916 | .8985 | .7720 | .8876 | 1.2954 | .1124 | 20 | 0.9134 |
| 0.6603 | 50 | .6134 | .7877 | .7898 | .8975 | .7766 | .8902 | 1.2876 | .1098 | 10 | 0.9105 |
| 0.6632 | 38° 00' | .6157 | 9.7893 | .7880 | 9.8965 | .7813 | 9.8928 | 1.2799 | 0.1072 | 52° 00' | 0.9076 |
| 0.6661 | 10 | .6180 | .7910 | .7862 | .8955 | .7860 | .8954 | 1.2723 | .1046 | 50 | 0.9047 |
| 0.6690 | 20 | .6202 | .7926 | .7844 | .8945 | .7907 | .8980 | 1.2647 | .1020 | 40 | 0.9018 |
| 0.6720 | 30 | .6225 | .7941 | .7826 | .8935 | .7954 | .9006 | 1.2572 | .0994 | 30 | 0.8988 |
| 0.6749 | 40 | .6248 | .7957 | .7808 | .8925 | .8002 | .9032 | 1.2497 | .0968 | 20 | 0.8959 |
| 0.6778 | 50 | .6271 | .7973 | .7790 | .8915 | .8050 | .9058 | 1.2423 | .0942 | 10 | 0.8930 |
| 0.6807 | 39° 00' | .6293 | 9.7989 | .7771 | 9.8905 | .8098 | 9.9084 | 1.2349 | 0.0916 | 51° 00' | 0.8901 |
| 0.6836 | 10 | .6316 | .8004 | .7753 | .8895 | .8146 | .9110 | 1.2276 | .0890 | 50 | 0.8872 |
| 0.6865 | 20 | .6338 | .8020 | .7735 | .8884 | .8195 | .9135 | 1.2203 | .0865 | 40 | 0.8843 |
| 0.6894 | 30 | .6361 | .8035 | .7716 | .8874 | .8243 | .9161 | 1.2131 | .0839 | 30 | 0.8814 |
| 0.6923 | 40 | .6383 | .8050 | .7698 | .8864 | .8292 | .9187 | 1.2059 | .0813 | 20 | 0.8785 |
| 0.6952 | 50 | .6406 | .8066 | .7679 | .8853 | .8342 | .9212 | 1.1988 | .0788 | 10 | 0.8756 |
| 0.6981 | 40° 00' | .6428 | 9.8081 | .7660 | 9.8843 | .8391 | 9.9238 | 1.1918 | 0.0762 | 50° 00' | 0.8727 |
| 0.7010 | 10 | .6450 | .8096 | .7642 | .8832 | .8441 | .9264 | 1.1847 | .0736 | 50 | 0.8698 |
| 0.7039 | 20 | .6472 | .8111 | .7623 | .8821 | .8491 | .9289 | 1.1778 | .0711 | 40 | 0.8668 |
| 0.7069 | 30 | .6494 | .8125 | .7604 | .8810 | .8541 | .9315 | 1.1708 | .0685 | 30 | 0.8639 |
| 0.7098 | 40 | .6517 | .8140 | .7585 | .8800 | .8591 | .9341 | 1.1640 | .0659 | 20 | 0.8610 |
| 0.7127 | 50 | .6539 | .8155 | .7566 | .8789 | .8642 | .9366 | 1.1571 | .0634 | 10 | 0.8581 |
| 0.7156 | 41° 00' | .6561 | 9.8169 | .7547 | 9.8778 | .8693 | 9.9392 | 1.1504 | 0.0608 | 49° 00' | 0.8552 |
| 0.7185 | 10 | .6583 | .8184 | .7528 | .8767 | .8744 | .9417 | 1.1436 | .0583 | 50 | 0.8523 |
| 0.7214 | 20 | .6604 | .8198 | .7509 | .8756 | .8796 | .9443 | 1.1369 | .0557 | 40 | 0.8494 |
| 0.7243 | 30 | .6626 | .8213 | .7490 | .8745 | .8847 | .9468 | 1.1303 | .0532 | 30 | 0.8465 |
| 0.7272 | 40 | .6648 | .8227 | .7470 | .8733 | .8899 | .9494 | 1.1237 | .0506 | 20 | 0.8436 |
| 0.7301 | 50 | .6670 | .8241 | .7451 | .8722 | .8952 | .9519 | 1.1171 | .0481 | 10 | 0.8407 |
| 0.7330 | 42° 00' | .6691 | 9.8255 | .7431 | 9.8711 | .9004 | 9.9544 | 1.1106 | 0.0456 | 48° 00' | 0.8378 |
| 0.7359 | 10 | .6713 | .8269 | .7412 | .8699 | .9057 | .9570 | 1.1041 | .0430 | 50 | 0.8348 |
| 0.7389 | 20 | .6734 | .8283 | .7392 | .8688 | .9110 | .9595 | 1.0977 | .0405 | 40 | 0.8319 |
| 0.7418 | 30 | .6756 | .8297 | .7373 | .8676 | .9163 | .9621 | 1.0913 | .0379 | 30 | 0.8290 |
| 0.7447 | 40 | .6777 | .8311 | .7353 | .8665 | .9217 | .9646 | 1.0850 | .0354 | 20 | 0.8261 |
| 0.7476 | 50 | .6799 | .8324 | .7333 | .8653 | .9271 | .9671 | 1.0786 | .0329 | 10 | 0.8232 |
| 0.7505 | 43° 00' | .6820 | 9.8338 | .7314 | 9.8641 | .9325 | 9.9697 | 1.0724 | 0.0303 | 47° 00' | 0.8203 |
| 0.7534 | 10 | .6841 | .8351 | .7294 | .8629 | .9380 | .9722 | 1.0661 | .0278 | 50 | 0.8174 |
| 0.7563 | 20 | .6862 | .8365 | .7274 | .8618 | .9435 | .9747 | 1.0599 | .0253 | 40 | 0.8145 |
| 0.7592 | 30 | .6884 | .8378 | .7254 | .8606 | .9490 | .9772 | 1.0538 | .0228 | 30 | 0.8116 |
| 0.7621 | 40 | .6905 | .8391 | .7234 | .8594 | .9545 | .9798 | 1.0477 | .0202 | 20 | 0.8087 |
| 0.7650 | 50 | .6926 | .8405 | .7214 | .8582 | .9601 | .9823 | 1.0416 | .0177 | 10 | 0.8058 |
| 0.7679 | 44° 00' | .6947 | 9.8418 | .7193 | 9.8569 | .9657 | 9.9848 | 1.0355 | 0.0152 | 46° 00' | 0.8029 |
| 0.7709 | 10 | .6967 | .8431 | .7173 | .8557 | .9713 | .9874 | 1.0295 | .0126 | 50 | 0.7999 |
| 0.7738 | 20 | .6988 | .8444 | .7153 | .8545 | .9770 | .9899 | 1.0235 | .0101 | 40 | 0.7970 |
| 0.7767 | 30 | .7009 | .8457 | .7133 | .8532 | .9827 | .9924 | 1.0176 | .0076 | 30 | 0.7941 |
| 0.7796 | 40 | .7030 | .8469 | .7112 | .8520 | .9884 | .9949 | 1.0117 | .0051 | 20 | 0.7912 |
| 0.7825 | 50 | .7050 | .8482 | .7092 | .8507 | .9942 | .9975 | 1.0058 | .0025 | 10 | 0.7883 |
| 0.7854 | 45° 00' | .7071 | 9.8495 | .7071 | 9.8495 | 1.0000 | 0.0000 | 1.0000 | 0.0000 | 45° 00' | 0.7854 |
| | | Nat. | Log. | Nat. | Log. | Nat. | Log. | Nat. | Log. | | |
| | | COSINES. | | SINES. | | COTANGENTS. | | TANGENTS. | | DEGREES. | RADIANS. |

Equivalents of Radians in Degrees, Minutes, and Seconds of Arc.

| RADIANS. | EQUIVALENTS. | RADIANS. | EQUIVALENTS. |
|----------|---------------------------|----------|-------------------------------|
| 0.0001 | 0° 0' 20".6 or 0°.005730 | 0.0600 | 3° 26' 15".9 or 3°.437747 |
| 0.0002 | 0° 0' 41".3 or 0°.011459 | 0.0700 | 4° 0' 38".5 or 4°.010705 |
| 0.0003 | 0° 1' 01".9 or 0°.017189 | 0.0800 | 4° 35' 01".2 or 4°.583662 |
| 0.0004 | 0° 1' 22".5 or 0°.022918 | 0.0900 | 5° 9' 23".8 or 5°.156620 |
| 0.0005 | 0° 1' 43".1 or 0°.028648 | 0.1000 | 5° 43' 46".5 or 5°.729578 |
| 0.0006 | 0° 2' 03".8 or 0°.034377 | 0.2000 | 11° 27' 33".0 or 11°.459156 |
| 0.0007 | 0° 2' 24".4 or 0°.040107 | 0.3000 | 17° 11' 19".4 or 17°.188734 |
| 0.0008 | 0° 2' 45".0 or 0°.045837 | 0.4000 | 22° 55' 05".9 or 22°.918312 |
| 0.0009 | 0° 3' 05".6 or 0°.051566 | 0.5000 | 28° 38' 52".4 or 28°.647890 |
| 0.0010 | 0° 3' 26".3 or 0°.057296 | 0.6000 | 34° 22' 38".9 or 34°.377468 |
| 0.0020 | 0° 6' 52".5 or 0°.114592 | 0.7000 | 40° 6' 25".4 or 40°.107046 |
| 0.0030 | 0° 10' 18".8 or 0°.171887 | 0.8000 | 45° 50' 11".8 or 45°.836624 |
| 0.0040 | 0° 13' 45".1 or 0°.229183 | 0.9000 | 51° 33' 58".3 or 51°.566202 |
| 0.0050 | 0° 17' 11".3 or 0°.286479 | 1.0000 | 57° 17' 44".8 or 57°.295780 |
| 0.0060 | 0° 20' 37".6 or 0°.343775 | 2.0000 | 114° 35' 29".6 or 114°.591559 |
| 0.0070 | 0° 24' 03".9 or 0°.401070 | 3.0000 | 171° 53' 14".4 or 171°.887339 |
| 0.0080 | 0° 27' 30".1 or 0°.458366 | 4.0000 | 229° 10' 59".2 or 229°.183118 |
| 0.0090 | 0° 30' 56".4 or 0°.515662 | 5.0000 | 286° 28' 44".0 or 286°.478898 |
| 0.0100 | 0° 34' 22".6 or 0°.572958 | 6.0000 | 343° 46' 28".8 or 343°.774677 |
| 0.0200 | 1° 8' 45".3 or 1°.145916 | 7.0000 | 401° 4' 13".6 or 401°.070457 |
| 0.0300 | 1° 43' 07".9 or 1°.718873 | 8.0000 | 458° 21' 58".4 or 458°.366236 |
| 0.0400 | 2° 17' 30".6 or 2°.291831 | 9.0000 | 515° 39' 43".3 or 515°.662016 |
| 0.0500 | 2° 51' 53".2 or 2°.864789 | 10.0000 | 572° 57' 28".1 or 572°.957795 |

The Values in Circular Measure of Angles which are given in Degrees and Minutes.

| | | | | | | | | | |
|----|--------|-----|--------|-----|--------|-----|--------|------|--------|
| 1' | 0.0003 | 9' | 0.0026 | 3° | 0.0524 | 20° | 0.3491 | 100° | 1.7453 |
| 2' | 0.0006 | 10' | 0.0029 | 4° | 0.0698 | 30° | 0.5236 | 110° | 1.9199 |
| 3' | 0.0009 | 20' | 0.0058 | 5° | 0.0873 | 40° | 0.6981 | 120° | 2.0944 |
| 4' | 0.0012 | 30' | 0.0087 | 6° | 0.1047 | 50° | 0.8727 | 130° | 2.2689 |
| 5' | 0.0015 | 40' | 0.0116 | 7° | 0.1222 | 60° | 1.0472 | 140° | 2.4435 |
| 6' | 0.0017 | 50' | 0.0145 | 8° | 0.1396 | 70° | 1.2217 | 150° | 2.6180 |
| 7' | 0.0020 | 1° | 0.0175 | 9° | 0.1571 | 80° | 1.3963 | 160° | 2.7925 |
| 8' | 0.0023 | 2° | 0.0349 | 10° | 0.1745 | 90° | 1.5708 | 170° | 2.9671 |

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